

# Properties of Convergence and Oscillating Points of the Generalized Infinite Tetration Function

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The infinite tetration function, composed of an infinitely high tower of its base, is notable for its counterintuitive convergence interval. In this article, we explore the convergence properties of the generalized infinite tetration function, where the initial value in the iterative process defining the infinite tetration can be any arbitrary number. We start by examining the conditions for convergence within the convergence interval, then focus mainly on the properties of the oscillating points. This paper provides a novel proof of the uniqueness and existence of these oscillating points and examines the function’s convergence to them. Furthermore, we propose a direct relationship between the values of the oscillating points and  $x$  in the generalized infinite tetration function, using the two branches of the Lambert  $W$  function.

Keywords: infinite tetration function, infinite power tower function, fixed-point iteration, convergence, Lambert  $W$  function

## 1 Introduction

The infinite tetration function, also known as the infinite power tower function, takes the form

$$f(x) = x^{x^{x^{\dots}}}$$

which raises a base to its own power infinitely many times. Since the tetration is constructed downward, the function can also be recursively represented as

$$f(x) = \lim_{n \rightarrow \infty} y_n,$$

where  $y_0 = x$  and  $y_n = x^{y_{n-1}}$  for  $n = 1, 2, 3, \dots$ <sup>1</sup>.

Due to its infinite nature, the function might seem to diverge at first glance. However, this function actually converges within a specific interval, which was first proven to be  $[e^{-e}, e^{\frac{1}{e}}]$  by Euler<sup>2</sup>.

The convergence value is specifically given by a fixed point of the map  $y \mapsto x^y$ , which is the solution  $y^*$  to the equation  $x^{y^*} = y^*$ . Outside the lower bound of the convergence interval, the function exhibits an intriguing behavior in which it oscillates between different values during the iterative process; we denote these values as *oscillating points*.

The infinite power tower appears in many intriguing mathematical problems, such as finding values of  $x$  and  $y$  that satisfy the relation  $x^y = y^{x^3}$ . Also, the concepts of fixed points and the convergence of iterations toward them—which play a significant role in analyzing the function—have numerous applications, including the analysis of equilibrium stability in game theory and the study of the behavior of dynamical systems in physics<sup>4</sup>.

It is a well-known fact that the Lambert  $W$  function, denoted by  $W(z)$ , is closely related to the infinite tetration function. The Lambert  $W$  function, defined on the complex plane, is the multivalued inverse function of  $f(z) = ze^z$ , where  $z \in \mathbb{C}$ . This function is useful when representing the solutions of equations involving exponentials or logarithms that cannot be expressed using elementary functions, and it has proven its significance in various fields such as enzyme kinetics and astrophysics<sup>5,6</sup>. For real arguments, the Lambert  $W$  function has two real branches,  $W_0(x)$  and  $W_{-1}(x)$ , which arise from the fact that  $f(x) = xe^x$  is not injective. As we will see, these two branches can be used to analyze the convergence points of the infinite tetration function.

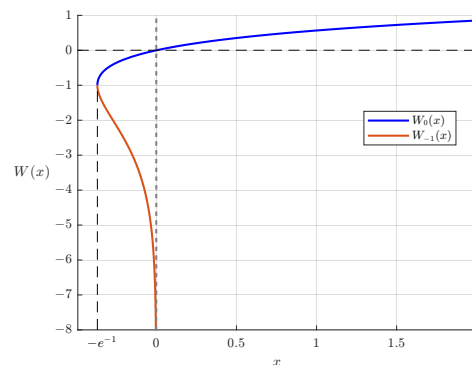


Fig. 1 Plot of the Lambert  $W$  function

In this article, we consider a modified form of the infinite tetration function that allows for an arbitrary initial value  $a$  in the iterative process:  $x^{\dots x^a}$ . We will refer to this modified function as the *generalized infinite tetration*. Toledo also examined

this function<sup>7</sup>, investigating its convergence depending on the value of  $x$  and the initial value  $a$ . This investigation is briefly demonstrated in the initial sections of this article, addressing the convergence interval of the generalized infinite tetration function and the representation of the convergence values using the Lambert  $W$  function. However, we would like to remind you that this paper focuses on the properties of oscillating points of the generalized infinite tetration function. Although some articles explain the oscillating points<sup>1,8</sup>, the direct relationship between  $x$  in the function and the oscillating points has not been determined. Thus, in this paper, we first demonstrate that only 2-cycles of oscillating points can exist, and that such cycles must uniquely exist for each fixed  $x$  in the function; that is, the same cycle will occur for a given  $x$  regardless of the choice of the initial value  $a$ . Next, we prove the convergence to these oscillating points using the fixed-point theorem, and finally, we represent the values of the oscillating points using the two branches of the Lambert  $W$  function:  $W_0(x)$  and  $W_{-1}(x)$ .

## 2 Convergence of the generalized infinite tetration function

The generalized infinite tetration function can alternatively be defined as the limit of an infinite sequence of exponentiations, as explained above. To determine whether this iterative sequence converges to a single value, we apply the fixed-point theorem, which provides the sufficient conditions for convergence of the fixed-point iteration.

**Lemma 1** (fixed-point theorem<sup>9</sup>). *If  $\alpha$  is a root of the equation  $x = f(x)$  where  $f(x)$  is a continuous and differentiable function, then the sequence of approximations  $\{x_n\}_{n \geq 0}$  by the fixed-point iteration will converge to the root  $\alpha$  provided the initial approximation  $x_0$  is chosen in  $I$ , where  $I$  is an interval containing the point  $x = \alpha$  and  $|f'(x)| < 1$  for all  $x \in I$ .*

*Proof.* Since  $\alpha$  is the root of  $x = f(x)$ ,  $\alpha = f(\alpha)$ . If  $x_{n-1}$  and  $x_n$  are two successive approximations to  $\alpha$ , then  $x_n = f(x_{n-1})$ . It can also be written as  $x_n - \alpha = f(x_{n-1}) - f(\alpha)$ . By the mean value theorem, there exists  $c \in (x_{n-1}, \alpha)$  such that  $\frac{f(x_{n-1}) - f(\alpha)}{x_{n-1} - \alpha} = f'(c)$ . Hence,  $x_n - \alpha = (x_{n-1} - \alpha)f'(c)$ . Let  $k$  be the supremum of  $|f'(x)|$  in  $I$ . Then,  $|f'(x)| \leq k < 1$ . Therefore,

$$|x_n - \alpha| = |x_{n-1} - \alpha| |f'(c)| \leq k |x_{n-1} - \alpha|.$$

Similarly,

$$|x_{n-1} - \alpha| \leq k |x_{n-2} - \alpha| \implies |x_n - \alpha| \leq k^2 |x_{n-2} - \alpha|.$$

Proceeding on,  $|x_n - \alpha| \leq k^n |x_0 - \alpha|$ . Since  $k < 1$ ,

$$0 \leq \lim_{n \rightarrow \infty} |x_n - \alpha| \leq \lim_{n \rightarrow \infty} k^n |x_0 - \alpha| = 0.$$

Hence,  $\lim_{n \rightarrow \infty} |x_n - \alpha| = 0$ , implying  $\lim_{n \rightarrow \infty} x_n = \alpha$ . Therefore,  $\{x_n\}$  converges to  $\alpha$ .  $\square$

**Definition 1** (The recursive definition of the generalized infinite tetration function). *Let  $y_0 = a$  where  $a \in \mathbb{R}$  and  $y_n = x^{y_{n-1}}$  for  $n = 1, 2, 3, \dots$ . The generalized infinite tetration function is defined as  $P(x, a) = \lim_{n \rightarrow \infty} y_n$ .*

Henceforth, we will denote the generalized infinite tetration function by  $P(x, a)$ . By Lemma 1, the convergence of  $P(x, a)$  to a stable fixed point requires that  $\left| \frac{d(x^y)}{dy} \right| < 1$  for a given  $x$ . Moroni<sup>1</sup> showed that, for the stable fixed point  $y$  satisfying  $y = x^y$ , the condition holds for  $y \in (e^{-1}, e)$ , and consequently,  $x \in (e^{-e}, e^{\frac{1}{e}})$ . It is further demonstrated that  $P(x, a)$  can converge even when  $\left| \frac{d(x^y)}{dy} \right| = 1$ , specifically at the boundary points  $(x, y) = (e^{-e}, e^{-1})$  and  $(e^{\frac{1}{e}}, e)$ . This is because  $f(y) = x^y$  has a "half stable" saddle fixed point when  $x = e^{\frac{1}{e}}$  and a unique stable fixed point when  $x = e^{-e}$ , which allows for convergence. Thus, it is established that  $P(x, a)$  can converge to some value in the interval  $[e^{-1}, e]$  when  $e^{-e} \leq x \leq e^{\frac{1}{e}}$ . The condition on the initial value  $a$  for  $P(x, a)$  to converge to this stable fixed point, as proved by Toledo<sup>7</sup>, is

- $\forall a \in \mathbb{R}$  if  $x \in [e^{-e}, 1]$
- $a < r$  if  $x \in (1, e^{\frac{1}{e}}]$

where  $r = \frac{W_{-1}(-\ln x)}{-\ln x}$  represents the largest real solution  $y$  of the equation  $y = x^y$  for  $x \in (1, e^{\frac{1}{e}}]$ .

## 3 Single-point convergence value of the infinite tetration function

Let  $k$  be the single-point convergence value of  $P(x, a)$  for  $x \in [e^{-e}, e^{\frac{1}{e}}]$ , then  $k = x^k$ . Applying the natural logarithm to both sides, we have the following.

$$\begin{aligned} \ln k &= k \ln x \\ \implies \ln x &= \frac{\ln k}{k} = e^{-\ln k} \ln k \\ \implies -\ln x &= e^{-\ln k} (-\ln k) \end{aligned}$$

Apply the Lambert  $W$  function to both sides of the equation.

$$W(-\ln x) = -\ln k$$

By the identity  $W(x) = \ln(x/W(x))$ , we have

$$k = e^{-W(-\ln x)} = e^{-\ln\left(\frac{-\ln x}{W(-\ln x)}\right)} = e^{\ln\left(\frac{W(-\ln x)}{-\ln x}\right)} = \frac{W(-\ln x)}{-\ln x}.$$

However, in the domain  $[-e^{-1}, 0)$ , the Lambert  $W$  function has two branches, namely  $W_0(x)$  and  $W_{-1}(x)$ , as shown in Figure 1. Note that  $-e^{-1} \leq -\ln x < 0$  implies  $1 < x \leq e^{\frac{1}{e}}$ . Thus, there are two possibilities,  $k = \frac{W_0(-\ln x)}{-\ln x}$  or  $k = \frac{W_{-1}(-\ln x)}{-\ln x}$ , that satisfy

the equation  $k = x^k$  for  $x$  in  $(1, e^{\frac{1}{e}}]$ .  $\frac{W_{-1}(-\ln x)}{-\ln x}$  is decreasing on  $(1, e^{\frac{1}{e}}]$ , since the denominator  $-\ln x$  is decreasing and negative and the numerator  $W_{-1}(-\ln x)$  is increasing and negative on that interval. Thus, the minimum value for  $\frac{W_{-1}(-\ln x)}{-\ln x}$  in  $(1, e^{\frac{1}{e}}]$  is

$$\left. \frac{W_{-1}(-\ln x)}{-\ln x} \right|_{x=e^{\frac{1}{e}}} = \frac{W_{-1}(-1/e)}{-1/e} = \frac{-1}{-1/e} = e.$$

In the previous section, it was shown that the stable fixed point  $y$  of  $P(x, a)$  should satisfy  $e^{-1} \leq y \leq e$ . Hence,  $\frac{W_{-1}(-\ln x)}{-\ln x}$  cannot be the stable fixed point except when  $x = e^{\frac{1}{e}}$ . However, the values of  $\frac{W_{-1}(-\ln x)}{-\ln x}$  and  $\frac{W_0(-\ln x)}{-\ln x}$  are equal when  $x = e^{\frac{1}{e}}$ . Therefore, the stable fixed point of  $P(x, a)$  generally has the form  $k = \frac{W_0(-\ln x)}{-\ln x}$  for  $x \in [e^{-e}, e^{\frac{1}{e}}]$ . It is also worth noting that a trivial case arises when the initial value  $a$  is equal to the unstable fixed point, i.e.,  $a = \frac{W_{-1}(-\ln x)}{-\ln x}$ . In this case,  $P(x, a)$  trivially takes the value  $\frac{W_{-1}(-\ln x)}{-\ln x}$ .<sup>7</sup>

When  $0 < x < e^{-e}$ ,  $P(x, a)$  does not converge but oscillates, and the iterative process seems to converge to several points. We will discuss more on this property in the following sections.

## 4 Uniqueness and existence of oscillating points

When  $x$  is outside the interval  $[e^{-e}, e^{\frac{1}{e}}]$ , the infinite tetration function does not converge. In particular, when  $x \in (0, e^{-e})$ , the function appears to converge to oscillating points, points that appear alternately with each iteration in Definition 1. However, first looking into this function, it is hard to infer how many true oscillating points there are. In this section, we will prove the uniqueness of oscillating points for each  $x$  in the infinite tetration function.

### 4.1 Nonexistence of $n$ -cycles of order greater than 2

Oscillating points form a cycle of distinct numbers for each fixed-point iteration. Let us assume that there is an  $n$ -cycle with  $n$  distinct oscillating points. Label its points in the order of iteration as a sequence  $(y_1, y_2, \dots, y_n)$  where  $y_{i+1} = x^{y_i}$  for  $i = 1, 2, \dots, n$  with indices taken mod  $n$  so that  $y_{n+1} = y_1$ . Let us examine whether any sets of oscillating points exist within the range of  $x > e^{\frac{1}{e}}$  or  $x < e^{-e}$ .

#### 1. $x > e^{\frac{1}{e}} > 1$

Since  $f(y) = x^y$  is strictly increasing, the map  $y \mapsto x^y$  is order-preserving. Applying this map thus has the effect of "shifting" every element in the ordered list one step. Now, suppose that the cycle contains at least two elements (i.e., it is nontrivial), and pick any two adjacent elements in the ordered list, say  $y_i$  and  $y_{i+1}$ . Since the elements are distinct,

either  $y_i < y_{i+1}$  or  $y_i > y_{i+1}$  holds. Then, we have

$$x^{y_i} < x^{y_{i+1}} \implies y_{i+1} < y_{i+2}$$

for the first case and

$$x^{y_i} > x^{y_{i+1}} \implies y_{i+1} > y_{i+2}$$

for the second case after applying the mapping once. We can repeat the same argument for  $y_{i+1}$  and  $y_{i+2}$ , and continue similarly for all adjacent pairs around the cycle in each case. Consequently, every element of the cycle is mapped strictly in the same direction, and we can never return to a previously visited value. Thus, no nontrivial cycle can be formed.

#### 2. $x < e^{-e} < 1$

Since  $f(y) = x^y$  is strictly decreasing,  $(f \circ f)(y) = x^{x^y}$  is strictly increasing and the map  $y \mapsto x^{x^y}$  is order-preserving. Thus, applying the similar "shifting" logic,  $y \mapsto x^{x^y}$  cannot have a nontrivial cycle. If any cycle exists, it must be trivial, i.e., an initial point  $y_i$  satisfies  $y_i = x^{x^{y_i}}$ .

Now, consider a nontrivial cycle under  $y \mapsto x^y$ . Then, under  $y \mapsto x^{x^y}$ , the sequence will visit every second point in the original cycle. This sequence of  $(y_1, y_3, y_5, \dots)$  eventually returns to  $y_1$ , specifically when  $2m \bmod n = 0$  where  $m$  is the number of iterations. Since  $y \mapsto x^{x^y}$  only has trivial cycles,  $m = 1$  must be a solution to that equation; that is,  $2 \bmod n = 0$ . The only positive integers satisfying this are  $n = 1$  and  $n = 2$ . Therefore, the oscillating points can have a period of at most 2.

### 4.2 Unique existence of 2-cycles

There is a possibility that depending on the initial value  $a$  of the iteration defining  $P(x, a)$ , the process could converge to different 2-cycles. Thus, it is reasonable to consider the existence of several distinct 2-cycles. We will now prove that a 2-cycle must exist uniquely for each  $x$  in the interval  $(0, e^{-e})$ . Let  $(y_1, y_2)$  be a pair of oscillating points in the same 2-cycle, where we can assume  $y_1 < y_2$  without loss of generality. We need to prove that there exists a unique pair  $(y_1, y_2)$  for each  $x$  in the interval  $(0, e^{-e})$  which satisfies the two following conditions:

$$\begin{cases} y_2 = x^{y_1} \\ y_1 = x^{y_2} \end{cases}$$

We observe that  $y_1, y_2 < 1$  since  $x < 1$ . Combining the two equations gives us

$$y_1^{y_1} = y_2^{y_2} \tag{1}$$

Let  $y_2 = py_1$  where  $p > 1$ , since  $y_1 < y_2$ . Apply the natural logarithm to both sides of Equation (1).

$$y_1 \ln y_1 = y_2 \ln y_2 = py_1 \ln(py_1)$$

$$\begin{aligned} &\implies \ln y_1 = p(\ln p + \ln y_1) \\ &\implies (p-1)\ln y_1 = -p \ln p \\ &\implies \begin{cases} y_1 = p^{\frac{p}{1-p}} \\ y_2 = p^{\frac{1}{1-p}} \end{cases} \end{aligned} \quad (2)$$

This representation of  $y_1$  and  $y_2$  using  $p$  is also proved by Moroni<sup>1</sup>. Now, apply  $y_2 = x^{y_1}$  on Equation (1).

$$y_1^{y_1} = x^{y_1 y_2}$$

We can rearrange this equation in terms of  $\ln x$ .

$$\ln x = \frac{y_1 \ln y_1}{y_1 y_2} = \frac{\ln y_1}{p y_1} = \frac{p}{1-p} \ln p \cdot \frac{1}{(p \cdot p^{\frac{p}{1-p}})} = \frac{\ln p \cdot p^{\frac{p}{1-p}}}{1-p}$$

It is sufficient to show that a unique  $p$  exists for each  $x$  to prove the unique existence of  $(y_1, y_2)$  for each  $x$ , since the value of  $y_1$  and  $y_2$  can be uniquely determined by the unique value of  $p$ , as shown by Equation (2). For  $0 < x < e^{-e}$ ,  $\ln x$  is strictly monotone with the range of  $-\infty < \ln x < -e$ . If  $\frac{\ln p \cdot p^{\frac{p}{1-p}}}{1-p}$  is also strictly monotone for  $p > 1$  and has the range containing  $\{y \mid -\infty < y < -e\}$ , then there must be one unique  $p$  for each fixed  $x$  in  $0 < x < e^{-e}$  that satisfies  $\ln x = \frac{\ln p \cdot p^{\frac{p}{1-p}}}{1-p}$ .

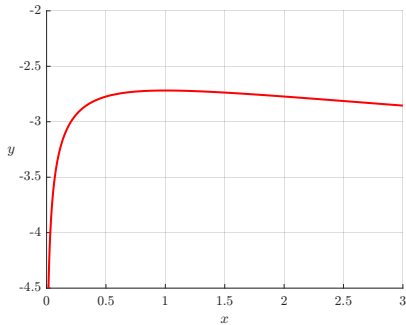


Fig. 2 Plot of  $y = \frac{\ln x \cdot x^{\frac{x}{1-x}}}{1-x}$

**Theorem 1.** The function  $f(x) = \frac{\ln x \cdot x^{\frac{x}{x-1}}}{1-x}$  is strictly monotone, and has the range  $\{y \mid -\infty < y < -e\}$  for  $x > 1$ .

*Proof.* We will first prove the monotonicity of the function, and then determine its range. Let  $x = e^{2t}$ , where  $t > 0$ . Then we have

$$\begin{aligned} \frac{\ln x \cdot x^{\frac{x}{x-1}}}{1-x} &= \frac{2t}{1-e^{2t}} e^{\frac{2te^{2t}}{e^{2t}-1}} = \frac{2t}{1-e^{2t}} e^{t \frac{2e^t}{e^t-e^{-t}}} = \frac{2t}{1-e^{2t}} e^{t \left(1 + \frac{e^t+e^{-t}}{e^t-e^{-t}}\right)} \\ &= \frac{2te^t}{1-e^{2t}} e^{t \frac{(e^t+e^{-t})/2}{(e^t-e^{-t})/2}} = \frac{t}{(e^{-t}-e^t)/2} e^{\frac{t \cosh(t)}{\sinh(t)}} \\ &= -\frac{t}{\sinh(t)} e^{\frac{t \cosh(t)}{\sinh(t)}} = -t \operatorname{csch}(t) e^{t \coth(t)}. \end{aligned}$$

Since  $x = e^{2t}$  is strictly monotone for  $x > 1$  and  $t > 0$ , it suffices to show that  $-t \operatorname{csch}(t) e^{t \coth(t)}$  is strictly monotone to prove that  $\frac{\ln x \cdot x^{\frac{x}{x-1}}}{1-x}$  is strictly monotone. Thus, We will now show that  $-t \operatorname{csch}(t) \cdot e^{t \coth(t)}$  is strictly monotone for  $t > 0$ . Differentiating the function with respect to  $t$ , we have

$$\frac{d}{dt}(-t \operatorname{csch}(t) e^{t \coth(t)}) = e^{t \coth(t)} \operatorname{csch}(t) (t^2 \operatorname{csch}^2(t) - 1).$$

Since  $t^2 \operatorname{csch}^2(t) = \frac{t^2}{\sinh^2(t)} < 1$  and  $\operatorname{csch}(t) > 0$  and  $e^{t \coth(t)} > 0$  holds for all  $t > 0$ , the following also holds for all  $t > 0$ .

$$\frac{d}{dt}(-t \operatorname{csch}(t) e^{t \coth(t)}) < 0$$

Therefore,  $-t \operatorname{csch}(t) e^{t \coth(t)}$  is strictly decreasing for  $t > 0$ , which proves that  $\frac{\ln x \cdot x^{\frac{x}{x-1}}}{1-x}$  is also strictly decreasing for  $x > 1$ .



Fig. 3 Plot of  $y = -x \operatorname{csch}(x) e^{x \coth(x)}$

Now we will determine the range of the function for  $x > 1$  by finding the value of the limit of  $f(x)$  when  $x$  approaches 1 and  $\infty$ .

1. The supremum of  $f(x)$  is the value of the limit of  $f(x)$  when  $x$  approaches 1.

$$\begin{aligned} \lim_{x \rightarrow 1} \left( \frac{\ln x \cdot x^{\frac{x}{x-1}}}{1-x} \right) &= \lim_{t \rightarrow 0} \left( -t \operatorname{csch}(t) e^{t \coth(t)} \right) \\ &= \lim_{t \rightarrow 0} \left( \frac{-t}{\sinh(t)} e^{t \coth(t)} \right) \\ &= \lim_{t \rightarrow 0} \left( \frac{-t}{\sinh(t)} e^{\frac{t}{\sinh(t)} \cosh(t)} \right) \end{aligned}$$

$$\text{Since } \lim_{t \rightarrow 0} \left( \frac{t}{\sinh(t)} \right) \stackrel{\text{L'Hôpital}}{=} \lim_{t \rightarrow 0} \left( \frac{1}{\cosh(t)} \right) = 1,$$

$$\lim_{t \rightarrow 0} \left( \frac{-t}{\sinh(t)} e^{\frac{t}{\sinh(t)} \cosh(t)} \right) = -e.$$

Thus, the supremum of  $f(x)$  is  $-e$ .

2. The infimum of  $f(x)$  is the value of the limit of  $f(x)$  when  $x$  approaches  $\infty$ .

$$\begin{aligned} \lim_{x \rightarrow \infty} \left( \frac{\ln x \cdot x^{x/(x-1)}}{1-x} \right) &= \lim_{t \rightarrow \infty} \left( -t \cdot \operatorname{csch}(t) \cdot e^{t \coth(t)} \right) \\ &= \lim_{t \rightarrow \infty} \left( -\frac{2te^{t \coth(t)}}{e^t - e^{-t}} \right) \\ &= \lim_{t \rightarrow \infty} \left( -\frac{2te^{t(\coth(t)-1)}}{1 - e^{-2t}} \right) = \lim_{t \rightarrow \infty} (-2t) = -\infty \end{aligned}$$

Thus, the infimum of  $f(x)$  is  $-\infty$ .

We can now conclude the following for  $x < 1$ :  $-\infty < \frac{\ln(x) \cdot x^{x/(x-1)}}{1-x} < -e$ . □

The proof ensures that  $\ln x = \frac{\ln p \cdot p^{p/(p-1)}}{1-p}$  must have one unique solution of  $p$  ( $p > 1$ ) for each fixed value of  $x$  in  $0 < x < e^{-e}$ . Therefore, a unique 2-cycle of oscillating points  $(y_1, y_2)$  exist for each  $x$  in  $0 < x < e^{-e}$ .

## 5 Convergence to oscillating points

In the previous section, we have shown that only a unique 2-cycle of oscillating points exists for each  $x$  in  $0 < x < e^{-e}$ . However, we also need to show that the iteration defining  $P(x, a)$  converges to this 2-cycle of oscillation points.

### 5.1 Condition on oscillating points

The pair of oscillating points  $(y_1, y_2)$  satisfies  $x^{y_1} = y_2$  and  $x^{y_2} = y_1$ , that is,  $x^{y_1} = y_1$  and  $x^{y_2} = y_2$ . By Lemma 1,  $\left| \frac{d(x^{y_1})}{dy_1} \right| < 1$  and  $\left| \frac{d(x^{y_2})}{dy_2} \right| < 1$  are sufficient conditions for the generalized infinite tetration function (and also for the infinite tetration function) to converge to the 2-cycle of oscillating points  $(y_1, y_2)$ . Computing the derivative, we have

$$\frac{d(x^{y_1})}{dy_1} = x^{y_1} \cdot x^{y_1} (\ln x)^2 = y_1 y_2 (\ln x)^2.$$

Since  $x^{y_2} = y_1$ ,

$$y_2 \ln x = \ln y_1 \implies \ln x = \frac{\ln y_1}{y_2}.$$

Thus,

$$\frac{d(x^{y_1})}{dy_1} = y_1 y_2 \left( \frac{\ln y_1}{y_2} \right)^2 = \frac{y_1 (\ln y_1)^2}{y_2}.$$

Let  $y_2 > y_1$  and  $y_2 = py_1$  where  $p > 1$ . Using the relation in Equation (2), we have

$$\frac{d(x^{y_1})}{dy_1} = \frac{y_1 (\ln y_1)^2}{y_2} = \frac{(\ln p)^2 p}{(1-p)^2}.$$

Let  $p = e^{2t}$  ( $t > 0$ ), then

$$\frac{d(x^{y_1})}{dy_1} = \frac{(\ln p)^2 p}{(1-p)^2} = \frac{e^{2t} (4t^2)}{(1-e^{2t})^2} = \frac{t^2}{\sinh^2(t)}.$$

Since  $t < \sinh t$  for  $t > 0$ ,

$$0 < \frac{d(x^{y_1})}{dy_1} = \frac{t^2}{\sinh^2(t)} < 1.$$

Thus,  $\left| \frac{d(x^{y_1})}{dy_1} \right| < 1$  holds when oscillating points exist. Similarly,

it can be shown that  $\left| \frac{d(x^{y_2})}{dy_2} \right| < 1$ . Therefore,  $y_1$  and  $y_2$  are stable fixed points of  $f(y) = x^{y^y}$ , and  $P(x, a)$  can converge to the 2-cycle of oscillating points  $(y_1, y_2)$ .

### 5.2 Condition on initial value of iteration

Now, we will conduct a graphical analysis to identify the conditions on the initial value  $a$  that result in  $P(x, a)$  converging to the 2-cycle of oscillating points for any fixed  $x$  in  $(0, e^{-e})$ . For the sake of convenience, we define the following functions:  $f_1(y) = x^{x^y}$ ,  $f_2(y) = x^y$ , and  $f_3(y) = y$ .

Consider the equation  $y = x^{y^y}$ . It is evident that the equation admits a trivial solution  $\beta$  satisfying  $\beta = x^\beta$ , since this implies  $x^{x^\beta} = x^\beta = \beta$ . Having previously established the unique existence of the 2-cycle of oscillating points, it follows that there are exactly two distinct solutions other than the trivial one. Denote these two nontrivial solutions by  $\alpha$  and  $\gamma$  (previously labeled  $y_1$  and  $y_2$ ), satisfying  $\alpha = x^\gamma$  and  $\gamma = x^\alpha$ . Without loss of generality, assume  $\alpha < \gamma$ . Since  $\beta$  is distinct from  $\gamma$ , either  $\beta > \gamma$  or  $\beta < \gamma$  must hold. The map  $y \mapsto x^y$  is order-reversing, so

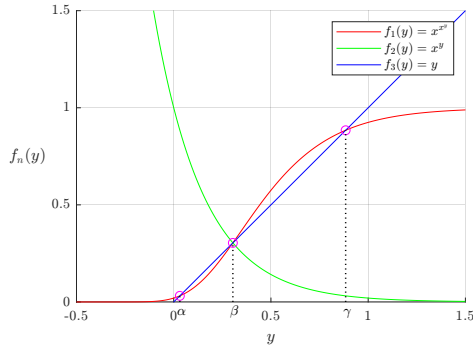
$$\beta > \gamma \implies x^\beta < x^\gamma \implies \beta < \alpha,$$

which contradicts the assumption  $\alpha < \gamma$ . On the other hand,

$$\beta < \gamma \implies x^\beta > x^\gamma \implies \beta > \alpha,$$

and it follows that  $\alpha < \beta < \gamma$ , that is, the trivial solution  $\beta$  must lie strictly between the two nontrivial solutions. Accordingly, we have three points of intersection between  $f_1(y)$  and  $f_3(y)$ , with the middle point also lying on  $f_2(y)$ , as shown in Figure 4.

Let us define the second value of the iteration in Definition 1 as  $b$ , where  $b = x^a$ . It is important to note that  $P(x, a)$  converges to the 2-cycle of oscillating points when the fixed-point iteration on the map  $y \mapsto f_1(y)$  with the initial value of  $a$  converges to one point, while the same iteration with the initial value  $b$  converges a different point. To check this condition, we examine the limit of convergence for different ranges of the initial value  $a_0$ . Before doing so, we first analyze some key properties of  $f_1(y)$ .



**Fig. 4** Plot of  $f_1(y) = x^{x^y}$ ,  $f_2(y) = x^y$ , and  $f_3(y) = y$  for  $x \in (0, e^{-e})$

The first derivative of  $f_1(y)$  is given by

$$f_1'(y) = \frac{d(x^{x^y})}{dy} = x^{x^y+y} \cdot (\ln x)^2,$$

so  $f_1'(y) > 0$  for all  $y \in \mathbb{R}$  whenever  $x \in (0, e^{-e})$ . Hence,  $f_1(y)$  is strictly increasing on  $\mathbb{R}$ . To determine the inflection point of  $f_1(y)$ , we compute the second derivative:

$$f_1''(y) = \frac{d}{dy} \left( x^{x^y+y} \cdot (\ln x)^2 \right) = x^{x^y+y} \cdot (x^y \ln x + 1) \cdot (\ln x)^3.$$

Setting  $f_1''(y) = 0$ , the only solution to this equation comes from the factor  $x^y \ln x + 1 = 0$ , which yields the unique inflection point at  $y = -\log_x(-\ln x)$ . According to the mean value theorem, there exists  $c_1 \in (\alpha, \beta)$  such that

$$f_1'(c_1) = \frac{f_1(\beta) - f_1(\alpha)}{\beta - \alpha} = \frac{\beta - \alpha}{\beta - \alpha} = 1$$

and there exists  $c_2 \in (\beta, \gamma)$  such that

$$f_1'(c_2) = \frac{f_1(\gamma) - f_1(\beta)}{\gamma - \beta} = \frac{\gamma - \beta}{\gamma - \beta} = 1.$$

Since  $f_1'(c_1) = f_1'(c_2) = 1$ , Rolle's theorem implies the existence of  $c \in (c_1, c_2) \subset (\alpha, \gamma)$  such that  $f_1''(c) = 0$ . It follows that  $c = -\log_x(-\ln x)$ , so the unique inflection point lies in the interval  $(\alpha, \gamma)$ . Observing  $f_1''(y)$ , we see that  $f_1''(y) > 0$  for  $y < c$  and  $f_1''(y) < 0$  for  $y > c$ .

**Case 1:**  $a_0 \geq \gamma$

For  $y > \gamma > c$ , we have  $f_1''(y) < 0$ , so  $f_1'(y)$  is decreasing. Since  $f_1'(y) > 0$  for all  $y \in \mathbb{R}$  and  $f_1'(\gamma) < 1$ , it follows that

$$0 < f_1'(y) \leq f_1'(\gamma) < 1$$

for all  $y \geq \gamma$ . Therefore, according to Lemma 1, the fixed-point iteration starting from  $a_0 \geq \gamma$  converges to  $\gamma$ .

**Case 2:**  $\beta < a_0 < \gamma$

Let  $\{a_n\}_{n \geq 0}$  be the sequence defined by  $a_{n+1} = f_1(a_n)$  with initial value  $\beta < a_0 < \gamma$ . Since  $f_1(y) > y$  for all  $y \in (\beta, \gamma)$ , we have

$a_{n+1} = f_1(a_n) > a_n$ , so the sequence is strictly increasing. Also, we can show that  $\{a_n\}$  is bounded above by  $\gamma$  using induction on  $n$ :  $a_0 < \gamma$  by the assumption and

$$a_n < \gamma \implies f_1(a_n) < f_1(\gamma) \implies a_{n+1} < \gamma$$

since  $f_1(y)$  is strictly increasing. Thus, by the monotone convergence theorem,  $\{a_n\}$  converges to its supremum,  $L = \sup_n a_n$ . Since  $f_1(y)$  is continuous and increasing,

$$f_1(L) = f_1\left(\sup_n a_n\right) = \sup_n f_1(a_n) = \sup_n a_{n+1} = L,$$

so  $L$  is a fixed point of  $f_1(y)$ . Since  $\beta < a_0 < L$ , the only fixed point that satisfies this is  $\gamma$ . Therefore,  $L = \gamma$ , and the fixed-point iteration starting from  $\beta < a_0 < \gamma$  converges to  $\gamma$ .

**Case 3:**  $a_0 = \beta$

It is evident that this fixed-point iteration converges to  $\beta$ , since  $\beta$  is a fixed point of  $f_1(y)$ .

**Case 4:**  $\alpha < a_0 < \beta$

Analogous to Case 2, the fixed-point iteration starting from  $\alpha < a_0 < \beta$  converges to  $\alpha$ .

**Case 5:**  $a_0 \leq \alpha$

For  $y < \alpha < c$ , we have  $f_1''(y) > 0$ , so  $f_1'(y)$  is increasing. Similarly to Case 1, it follows that

$$0 < f_1'(y) \leq f_1'(\alpha) < 1$$

for all  $y \leq \alpha$ . Therefore, the fixed-point iteration starting from  $a_0 < \alpha$  converges to  $\alpha$ .

Combining the results for all 5 cases, we can conclude that the fixed-point iteration on  $y \mapsto f_1(y)$  with the initial point  $a_0$  converges to

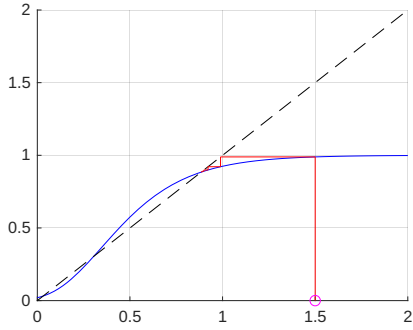
$$\begin{cases} \alpha & \text{if } a_0 < \beta, \\ \beta & \text{if } a_0 = \beta, \\ \gamma & \text{if } a_0 > \beta. \end{cases}$$

This repulsive property of the trivial fixed point  $y = \beta$  can be further explained by evaluating the derivative of  $f_1(y)$  at that point. Note that  $f_1(y) = x^{x^y}$  exhibits asymptotic behavior: for  $x < e^{-e} < 1$ ,  $\lim_{y \rightarrow \infty} f_1(y) = 1$  and  $\lim_{y \rightarrow -\infty} f_1(y) = 0$ . Given this behavior and the uniqueness of the inflection point of  $f_1(y)$ , one can obtain that the  $f_1(y)$  cannot be tangent to  $f_3(y) = y$  and must intersect it transversely at each fixed point. Thus,  $f_1(y) - f_3(y)$  changes sign from negative to positive at  $y = \beta$ , and from positive to negative at  $y = \alpha$  and  $y = \gamma$ . Therefore,

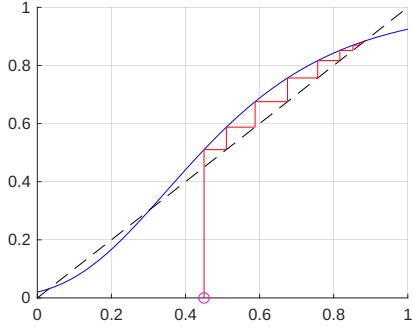
$$f_1'(\beta) - f_3'(\beta) > 0 \implies f_1'(\beta) > f_3'(\beta) = 1,$$

so  $y = \beta$  is an unstable fixed point.

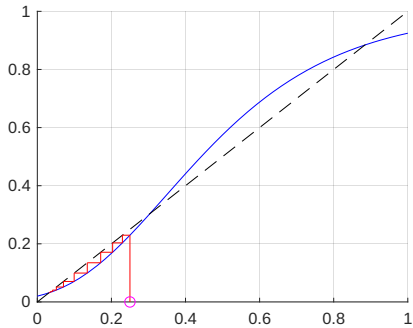
Now, returning to the original question, if  $a = \beta$ , then  $b = x^\beta = \beta$ , so the fixed-point iterations on  $y \mapsto f_1(y)$  starting from  $a$  and  $b$  both converge to the fixed point  $\beta$ . In fact,  $\beta = \frac{W_0(-\ln x)}{-\ln x}$ ,



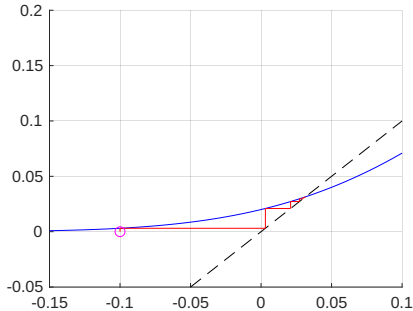
$a_0 > \gamma$ : converges to  $\gamma$



$\beta < a_0 < \gamma$ : converges to  $\gamma$



$\alpha < a_0 < \beta$ : converges to  $\alpha$



$a_0 < \alpha$ : converges to  $\alpha$

**Fig. 5** Cobweb plot of the fixed-point iteration on  $f_1(y)$  with the initial value  $a_0$

which can be derived using the method presented in the previous sections. In this case, oscillating points do not exist, but  $P(x, \beta)$  rather converges to a single value, namely  $\beta = \frac{W_0(-\ln x)}{-\ln x}$ . In other cases, however, we have either

$$a > \beta \implies x^a < x^\beta \implies b < \beta$$

or

$$a < \beta \implies x^a > x^\beta \implies b > \beta,$$

so each iteration starting from  $a$  and  $b$  lies on opposite sides of  $\beta$ . Thus, the two iterations converge to separate fixed points, namely  $\alpha$  and  $\gamma$ .

In summary, for each fixed  $x$  in  $(0, e^{-e})$ ,  $P(x, a)$  converges to the 2-cycle of oscillating points  $(y_1, y_2) = (\alpha, \gamma)$  for all  $a \in \mathbb{R}$  except when  $a = \beta$ . In that case, the function converges to a single point and  $P(x, \beta) = \beta$  holds for  $x$  in  $(0, e^{-e})$ .

## 6 Values of oscillating points

In this section, we will find the direct relation between the oscillating points  $(y_1, y_2)$  and  $x$  of the infinite tetration function. We can determine the limits of  $y_1$  and  $y_2$  as  $p$  approaches  $\infty$  and 1, by applying the natural logarithm to both sides of the equations in (2).

$$\lim_{p \rightarrow \infty} \ln y_1 = \lim_{p \rightarrow \infty} \frac{p}{1-p} \ln p = -\infty \implies \lim_{p \rightarrow \infty} y_1 = 0$$

$$\lim_{p \rightarrow \infty} \ln y_2 = \lim_{p \rightarrow \infty} \frac{1}{1-p} \ln p = 0 \implies \lim_{p \rightarrow \infty} y_2 = 1$$

$$\lim_{p \rightarrow 1} \ln y_1 = \lim_{p \rightarrow 1} \frac{p}{1-p} \ln p = -1 \implies \lim_{p \rightarrow 1} y_1 = \frac{1}{e}$$

$$\lim_{p \rightarrow 1} \ln y_2 = \lim_{p \rightarrow 1} \frac{1}{1-p} \ln p = -1 \implies \lim_{p \rightarrow 1} y_2 = \frac{1}{e}$$

Since  $\frac{p}{1-p} \ln p$  and  $\frac{1}{1-p} \ln p$  are strictly decreasing and the natural logarithm is strictly increasing, the values of the limits are either the supremum or the infimum of the range of  $y_1$  or  $y_2$ . Thus, the range of  $y_1$  and  $y_2$  is

$$\begin{cases} 0 < y_1 < \frac{1}{e} \\ \frac{1}{e} < y_2 < 1 \end{cases} \quad (3)$$

Note that  $y_2 \neq 1$  holds, as mentioned in the section on the unique existence of 2-cycles, and  $y_1 \neq 0$  since  $y_1 = x^{y_2}$ . We can also verify that the limits of  $x$  as  $p$  approaches  $\infty$  and 1 correspond to the infimum and supremum of the range of  $x$  where oscillating points emerge.

$$\lim_{p \rightarrow 1} x = \lim_{p \rightarrow 1} y_1^{\frac{1}{y_2}} = \left(\frac{1}{e}\right)^e = e^{-e}$$

$$\lim_{p \rightarrow \infty} x = \lim_{p \rightarrow \infty} y_1^{\frac{1}{y_2}} = 0$$

We obtain the following relations by applying the natural logarithm to Equation (1) and using the identity  $x = e^{\ln x}$ .

$$y_1 \ln y_1 = y_2 \ln y_2 = \ln y_1 \cdot e^{\ln y_1} = \ln y_2 \cdot e^{\ln y_2}$$

Hence, we have

$$\ln y_1 \cdot e^{\ln y_1} = y_2 \ln y_2, \tag{4}$$

$$\ln y_2 \cdot e^{\ln y_2} = y_1 \ln y_1. \tag{5}$$

Since  $y_1$  and  $y_2$  are both in the interval  $(0, 1)$ , it follows that  $-1/e \leq y_1 \ln y_1 = y_2 \ln y_2 < 0$ . Thus,  $y_1 \ln y_1$  and  $y_2 \ln y_2$  are both in the domain of  $W_0(x)$  and  $W_{-1}(x)$  and we can apply the Lambert  $W$  function to Equation (4) and Equation (5). By the ranges of  $y_1$  and  $y_2$  in (3), we have  $-\infty < \ln y_1 < -1$  and  $-1 < \ln y_2 < 0$ . Since  $W_{-1}(x) \leq -1 \leq W_0(x)$ , we should apply  $W_{-1}(x)$  to Equation (4) and  $W_0(x)$  to Equation (5).

$$\ln y_1 = W_{-1}(y_2 \ln y_2) \implies y_1 = e^{W_{-1}(y_2 \ln y_2)} = \frac{y_2 \ln y_2}{W_{-1}(y_2 \ln y_2)}$$

$$\ln y_2 = W_0(y_1 \ln y_1) \implies y_2 = e^{W_0(y_1 \ln y_1)} = \frac{y_1 \ln y_1}{W_0(y_1 \ln y_1)}$$

The identity  $W(x) = \ln\left(\frac{x}{W(x)}\right)$  was used in the final step.

Since  $x^{y_2} = y_1$ , we obtain the following result:

$$\begin{aligned} \ln x &= \frac{\ln y_1}{y_2} = \frac{\ln y_1}{(y_1 \ln y_1)/(W_0(y_1 \ln y_1))} = \frac{W_0(y_1 \ln y_1)}{y_1} \\ \implies x &= e^{\frac{W_0(y_1 \ln y_1)}{y_1}} \end{aligned}$$

Similarly, using  $x^{y_1} = y_2$ , we have

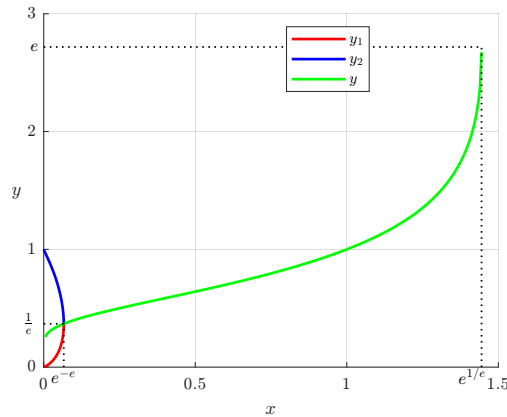
$$\begin{aligned} \ln x &= \frac{\ln y_2}{y_1} = \frac{\ln y_2}{(y_2 \ln y_2)/(W_{-1}(y_2 \ln y_2))} = \frac{W_{-1}(y_2 \ln y_2)}{y_2} \\ \implies x &= e^{\frac{W_{-1}(y_2 \ln y_2)}{y_2}} \end{aligned}$$

Therefore, although we cannot express  $y_1$  and  $y_2$  as functions of  $x$  using previously defined elementary or special functions, we can conclude that the following relation holds.

$$x = e^{\frac{W_0(y_1 \ln y_1)}{y_1}} = e^{\frac{W_{-1}(y_2 \ln y_2)}{y_2}} \tag{6}$$

## 7 Conclusion

In this article, we presented a novel proof for the properties of oscillating points in the generalized infinite tetration function, which are observed within the range  $0 < x < e^{-e}$ . Using the fixed-point theorem and other calculus techniques, we proved the uniqueness and existence of these oscillating points for each



**Fig. 6** Convergence behavior of  $P(x, a)$ :  $y_1$  and  $y_2$  indicate oscillating points, while  $y$  represents the single-point convergence value  $\frac{W_0(-\ln x)}{-\ln x}$

fixed  $x$ , and examined the function's convergence to them. Additionally, we conducted a graphical analysis to identify the conditions on the initial value required for convergence of the iteration. Finally, we derived a direct relationship between the values of the oscillating points and  $x$ , using the two branches of the Lambert  $W$  function. All in all, the convergence values and oscillating points of  $P(x, a)$  are given as follows:

$$P(x, a) = \begin{cases} \frac{W_0(-\ln x)}{-\ln x} & \text{if } \begin{cases} x \in (0, e^{-e}) \text{ and } a = \frac{W_0(-\ln x)}{-\ln x}, \\ \text{or } x \in [e^{-e}, 1] \text{ and } a \in \mathbb{R}, \\ \text{or } x \in (1, e^{\frac{1}{e}}] \text{ and } a < \frac{W_{-1}(-\ln x)}{-\ln x} \end{cases} \\ \frac{W_{-1}(-\ln x)}{-\ln x} & \text{if } x \in (1, e^{\frac{1}{e}}] \text{ and } a = \frac{W_{-1}(-\ln x)}{-\ln x} \\ (y_1, y_2) & \text{if } x \in (0, e^{-e}) \text{ and } a \neq \frac{W_0(-\ln x)}{-\ln x} \end{cases}$$

where  $y_1$  and  $y_2$  are given by Equation (6).

While our results establish the properties of the oscillating points and convergence conditions of the generalized infinite tetration function  $P(x, a)$ , they are limited to real inputs. Future work could further generalize the function by either extending the domain of the function to complex values of  $x$  and  $a$ , or by allowing more values in the power tower to be arbitrarily chosen. Investigating the possibility of cycles of order greater than 2 in these generalizations may provide a more comprehensive view of the oscillating points of the infinite tetration.

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