Positive Even Integer Values of the Riemann Zeta Function

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This paper was inspired by Euler's beautiful solution to the Basel Problem. In this paper we investigate further into the Riemann Zeta Function $\zeta(s)$ (In which the Basel Problem was a special case) and, in particular, its values at all positive even integer values. We begin by describing Herglotz's proof of the product formula for the sine function, and explain it for the reader in greater detail. We then combine this result with the Taylor Expansion of the sine function to show Euler's solution to the Basel Problem (The Zeta Funtion at the value s=2). This method of comparing coefficients at various powers of x can be adapted to find the solution to the ζ-function at other positive even integer values of s, and we provide examples using $\zeta(4)$ and $\zeta(6)$. We note that the effort to find these solutions become increasingly complicated as the value of s increase, so we recreate the derivation of the already-proven general formula for all $\zeta(2n), n \in \mathbb{Z}_{>0}$. To do so we will first give a short introduction to the sequence of rational numbers known as Bernoulli Numbers. Finally, in the last section we shall provide a new proof for the general formula, through mathematical induction using Newton's Identity. This reaffirms the result and further ensures its validity.

Keywords: The Zeta Function, Product Formula, Basel Problem, Bernoulli Numbers, Newton's Identity, Taylor Series, Newton's Identity, Bernhard Riemann, Leonhard Euler

Introduction

The Riemann Zeta Function originated from the Basel Problem, concerning the infinite sum of inverse squares posed by Pietro Mengoli in 1650. This problem was eventually solved by Leonhard Euler in 1734 using the Product Formula for the sine function. The zeta function is later defined by Bernhard Riemann in his 1859 paper On the Numer of Primes Less Than a Given Magnitude and extended the power of the reciprocals to other real values as well as complex values [1].

Definition 1. *The zeta function is defined as* $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ *for every complex number s with* $Re(s) > 1$.

A key problem involving the zeroes of the zeta function is the Riemann Hypothesis, proposed in 1859, concerning the distribution of complex zeroes, more specifically, whether or not the real part of every nontrivial zero of the function is $\frac{1}{2}$ [2].

There's been numerous results and conjectures regarding the values of the zeta function, such as at non-positive integers [3] and at complex values [4].

The general formula for the zeroes of the zeta function at positive even integer has been derived through several different methods, such as using the Fourier Series and Parseval's Identity [5]. In this paper we give a summary to the proof of the general formula through Euler's Cotangent Identiety, before introducing our new proof.

The paper proceeds as follows. In Section 2 we derive Taylor series for $\sin(x)$ and introduce a proof for the product formula for $\sin x$. In Section 3 we use the two results obtained in the previous section to calculate some particular values of the Riemann Zeta Function at even integers. We then derive a general formula for such values in Section 4 using Bernoulli Numbers. Our main result is in Section 5 where we prove this general formula via a more elementary approach by using mathematical induction.

The Taylor series and the product formula for $\sin(x)$

In this section we derive the Taylor series and the product formula for $\sin(x)$ so that we can later obtain the particular values $\zeta(n)$ for $n \in \mathbb{N}$ by comparing the coefficients at the different powers of x of the two expressions.

Theorem 2. *The Taylor series of* $\sin(x)$ *is*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Proof. The Taylor series [6] is defined as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

We define $f(x) = \sin x$ and work out its derivatives: $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$ and $f''''(x) = \sin x$. We can see that this is periodic so, taking a to be 0, we have $\sin(a) = 0$ and $\cos(a) = 1$ so that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and note that this is equal to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

□

Now we begin the proof[7] for the infinite product formula for the sine function,

$$\sin(x) = x \prod_{i=1}^{\infty} \left(1 - \frac{x^2}{i^2\pi^2}\right)$$

by proving an equivalent equation. Our exposition closely follows.

Proposition 3. *The infinite product formula for the sine function is equivalent to*

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{i=1}^{\infty} \left(\frac{1}{x+i} + \frac{1}{x-i}\right)$$

Proof. First we take logs of both sides. Using the identity $\ln \prod f(x) = \sum \ln f(x)$ we have

$$\ln\sin(x) = \ln x + \sum_{i=1}^{\infty} \ln\left(1 - \frac{x^2}{i^2\pi^2}\right)$$

Then differentiate both sides of this to get

$$\cot(x) = \frac{1}{x} - \sum_{i=1}^{\infty} \frac{2x}{i^2\pi^2 - x^2}$$

Replacing x by πx we get

$$\cot(\pi x) = \frac{1}{\pi x} - \sum_{i=1}^{\infty} \frac{2x}{\pi(i^2 - x^2)}$$

Multiply both sides by π and then turn $\frac{2x}{\pi(i^2-x^2)}$ into partial fractions to obtain

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{i=1}^{\infty} \left(\frac{1}{x-i} + \frac{1}{x+i}\right) \tag{1}$$

By integrating and then exponentiating this function, we obtain something similar to the product formula for the sine function, with difference in only the constant term. When we divide both sides by x we have

$$\frac{\sin(x)}{x} = \prod_{i=1}^{\infty} \left(1 - \frac{x^2}{i^2\pi^2}\right) e^C$$

Taking the limit $x \to 0$, both $\frac{\sin(x)}{x}$ and $\prod_{i=1}^{\infty}\left(1 - \frac{x^2}{i^2\pi^2}\right)$ approach 1 so we know that e^C must also be 1. This means that (1) is equivalent to the infinite product formula for the sine function.

□

We now need a few lemmas.

Lemma 4. $\lim_{x\to 0} \pi\cot(\pi x) - \frac{1}{x} = 0$

We simplify this to a single fraction, written in terms of $\cos(x)$ and $\sin(x)$, then using Taylor's Expansion we have

$$\pi\cot(\pi x) - \frac{1}{x} = \frac{\pi x\cos(\pi x) - \sin(\pi x)}{x\sin(\pi x)}$$

$$= \frac{\pi x(1 - \frac{(\pi x)^2}{2!} + \frac{(\pi x)^4}{4!} - \ldots) - (\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \ldots)}{x(\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \ldots)}$$

The numerator has the first x term at order 3 and the denominator at order 2. Thus the numerator approaches 0 faster than the denominator, so as $x \to 0$, $\pi\cot(\pi x) - \frac{1}{x} \to 0$.

Lemma 5. *Let* $f(x) = \pi\cot(\pi x)$ *and* $g(x) = \frac{1}{x} + \sum_{i=1}^{\infty}\left(\frac{1}{i-x} + \frac{1}{i+x}\right)$, *then*

$$f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) = 2f(x)$$

$$g\left(\frac{x}{2}\right) + g\left(\frac{x+1}{2}\right) = 2g(x)$$

Proof. To prove the identity for f, we write out the left side of the equation:

$$\pi\cot\frac{\pi x}{2} + \pi\cot\frac{\pi(x+1)}{2} = \frac{\cos\frac{\pi x}{2}\sin\frac{\pi(x+1)}{2} + \cos\frac{\pi(x+1)}{2}\sin\frac{\pi x}{2}}{\sin\frac{\pi x}{2}\sin\frac{\pi(x+1)}{2}}$$

We simplify this using the identities $\sin\frac{\pi(x+1)}{2} = \cos\frac{\pi x}{2}$; $\cos\frac{\pi(x+1)}{2} = -\sin\frac{\pi x}{2}$ and then the double-angle identities to obtain

$$f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) = \frac{\cos^2\frac{\pi x}{2} - \sin^2\frac{\pi x}{2}}{\sin\frac{\pi x}{2}\cos\frac{\pi x}{2}} = 2\pi\cot(\pi x) = 2f(x)$$

We move on to prove the identity for g.

We define $g_N(x) = \frac{1}{x} + \sum_{i=1}^{N}\left(\frac{1}{x+i} + \frac{1}{x-i}\right)$ to be the partial sum of the series. So when adding $g_N\left(\frac{x}{2}\right)$ and $g_N\left(\frac{x+1}{2}\right)$ we have

$$g_N\left(\frac{x}{2}\right) + g_N\left(\frac{x+1}{2}\right) =$$

$$2\left(\frac{1}{x} + \sum_{i=1}^{N}\left(\frac{1}{x+2i} + \frac{1}{x-2i}\right) + \frac{1}{x+1} + \sum_{i=1}^{N}\left(\frac{1}{x+2i+1} + \frac{1}{x-2i+1}\right)\right)$$

Now consider the different terms being summed:

$$\sum_{i=1}^{N}\frac{1}{x+2i}=\frac{1}{x+2}+\frac{1}{x+4}+\frac{1}{x+6}+...+\frac{1}{x+2N}$$

$$\sum_{i=1}^{N}\frac{1}{x-2i}=\frac{1}{x-2}+\frac{1}{x-4}+\frac{1}{x-6}+...+\frac{1}{x-2N}$$

$$\sum_{i=1}^{N}\frac{1}{x+2i+1}=\frac{1}{x+3}+\frac{1}{x+5}+\frac{1}{x+7}+...+\frac{1}{x+2N+1}$$

$$\sum_{i=1}^{N}\frac{1}{x-2i+1}=\frac{1}{x-1}+\frac{1}{x-3}+\frac{1}{x-5}+...+\frac{1}{x-2N+1}$$

Note that the sum of these terms plus $\frac{1}{x}$ and $\frac{1}{x+1}$ is equal to

$$\frac{1}{x}+\sum_{i=1}^{2N}\left(\frac{1}{x+i}+\frac{1}{x-i}\right)+\frac{1}{x+2N+1}$$

We must add the $\frac{1}{x+2N+1}$ term as it is outside of the boundary of the sum. Note that this is equal to $g_{2N}(x)+\frac{1}{x+2N+1}$. So we have $g_N\left(\frac{x}{2}\right)+g_N\left(\frac{x+1}{2}\right)=2g_{2N}(x)+\frac{2}{x+2N+1}$. Since we are calculating an infinite sum, $N\to\infty$ so both g_N and g_{2N} become g and $\frac{2}{x+2N+1}\to 0$. So we have

$$g\left(\frac{x}{2}\right)+g\left(\frac{x+1}{2}\right)=2g\left(x\right)$$

as required. □

Lemma 6. *The function h, defined by $h(x)=f(x)-g(x)$ for $x\in\mathbb{R}\setminus\mathbb{Z}$ and $h(x)=0$ for $x\in\mathbb{Z}$, is continuous, odd and has a period of 1.*

Proof. First note that h is clearly periodic and has period 1 since we defined it as 0 at each $x\in\mathbb{Z}$. We then prove that the function h is odd by proving that both f and h are odd. Indeed, we have

$$f(-x)=\pi\cot(-\pi x)=\frac{\pi\cos(-x)}{\sin(-x)}=-\pi\cot(\pi x)=-f(x)$$

since $\cos(-x)=\cos(x)$ and $\sin(-x)=-\sin(x)$.

$$g(-x)=-\frac{1}{x}-\sum_{i=1}^{\infty}\left(\frac{1}{x+i}+\frac{1}{x-i}\right)=-g(x)$$

The sum of two odd functions is also odd. So h must also be odd. Now we prove the continuity of h. At points $x\notin\mathbb{Z}$ it is clearly continuous. To prove its continuity at $x\in\mathbb{Z}$ we look at its limits.

$$h(x)=\pi\cot(\pi x)-\frac{1}{x}-\sum_{i=1}^{\infty}\left(\frac{1}{x+i}+\frac{1}{x-i}\right)$$

As $x\to 0$, $\pi\cot(\pi x)-\frac{1}{x}\to 0$ by lemma 4 and $\lim_{x\to 0}\sum_{i=1}^{\infty}\left(\frac{1}{x+i}+\frac{1}{x-i}\right)=\sum_{i=1}^{\infty}\lim_{x\to 0}\left(\frac{1}{x+i}+\frac{1}{x-i}\right)=0$. So we know that $\lim_{x\to 0}h(x)=0$ which means that it is continuous at $x=0$ and, by periodicity, is therefore continuous at every other $x\in\mathbb{Z}$. □

Now we can finally prove the main statement.

Theorem 7. *The infinite product formula for the sine function is*

$$\sin(x)=x\prod_{i=1}^{\infty}\left(1-\frac{x^2}{i^2\pi^2}\right)$$

Proof. We can prove this result by proving its equivalent,

$$\pi\cot(\pi x)=\frac{1}{x}+\sum_{i=1}^{\infty}\left(\frac{1}{x+i}+\frac{1}{x-i}\right).$$

This is proved if we can show that $h(x)=0$ for all x. From Lemma 6 h is periodic and continuous, which means it must have a maximum point, say M, at point x_0. The identity in lemma 5 for f and g clearly also applies for h. So we have

$$h\left(\frac{x_0}{2}\right)+h\left(\frac{x_0+1}{2}\right)=2h\left(x_0\right)=2M$$

Note that both $h\left(\frac{x_0}{2}\right)$ and $h\left(\frac{x_0+1}{2}\right)$ cannot be less than M since it implies that the other is greater than M which is not possible. This means that they are both equal to M. We can repeat this process for all $h\left(\frac{x_0}{2^n}\right)$. Then

$$M=\lim_{n\to\infty}h\left(\frac{x_0}{2^n}\right)=h(0)=0$$

by continuity of $h(x)$. This means that $h(x)\leq 0$ for all $x\in\mathbb{R}$. Since h is an odd function, $h(x)<0$ implies that $-h(x)=h(-x)>0$ which is not possible. So h must be equal to 0 for all x. □

Computing $\zeta(2)$, $\zeta(4)$, $\zeta(6)$

In this section we use our results in the previous section to compute particular values of the ζ function at $n=2,4,6$ by comparing coefficients. Note that the different coefficients of the product formula for the sine function have a structure $\sum_{i=1}^{\infty}\frac{1}{i^2\pi^2}$ and this is similar to the $\zeta(2)$ function.So we can obtain the value of the $\zeta(2)$ by finding the value of this coefficient via an equivalent formula (in this case we use the Taylor Expansion). Similarly, we can use this method to calculate any value of the ζ-function at positive even integers, since they can be decomposed into ζ-functions at lesser values (for example, $\zeta(4)$ can be expressed in terms of $\zeta(2)$. Note that this method will not work for any odd integers, since the product formula has a coefficient of 0 at any odd powers of x.

Proposition 8. *The value of $\zeta(2)$ is*

$$\zeta(2) = \frac{\pi^2}{6}$$

Proof. We consider the two equivalent series for $\sin(x)$, the product formula and the Taylor expansion. Dividing both sides by x gives

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ... = \prod_{i=1}^{\infty}\left(1 - \frac{x^2}{i^2\pi^2}\right) \tag{2}$$

To find $\zeta(2)$, we compare the coefficients of the x^2 terms in both sides of this equation. So

$$\frac{1}{3!} = \sum_{i=1}^{\infty} \frac{1}{i^2\pi^2}$$

We now multiply both sides by π, so the right hand side turns to $\sum_{i=1}^{\infty} \frac{1}{i^2}$ which is $\zeta(2)$ by definition. Hence $\zeta(2) = \frac{\pi^2}{6}$. □

Proposition 9. *The value of $\zeta(4)$ is*

$$\zeta(4) = \frac{\pi^4}{90}$$

Proof. We look again at (2) and compare the x^4 terms. In the expansion of $1 - \frac{x^2}{i^2\pi^2}$ we can view the x^2 terms as all possible combinations of products of two $\frac{x^2}{i^2\pi^2}$ terms from two brackets while the terms from the rest of the brackets being 1's. Then multiply both sides by π^4 to remove it from the denominator.

$$\frac{\pi^4}{5!} = \sum_{m>n\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^2}$$

Now consider the product of two series expansions for $\zeta(2)$ as follows

$$\zeta(2)^2 = \left(\sum_{m\geq 1} \frac{1}{m^2}\right)\left(\sum_{n\geq 1} \frac{1}{n^2}\right).$$

That is equal to the sum over all the cases where $m > n$ plus the cases where $n > m$ plus the cases where they are equal. Hence

$$\left(\sum_{m\geq 1} \frac{1}{m^2}\right)\left(\sum_{n\geq 1} \frac{1}{n^2}\right) = \sum_{m>n\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^2} + \sum_{n>m\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^2} + \sum_{n\geq 1}^{\infty} \frac{1}{n^4}$$

Rewriting this in terms of values of the ζ-function gives

$$\zeta(2)^2 = \sum_{m>n\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^2} + \sum_{n>m\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^2} + \zeta(4)$$

Since we already know $\zeta(2) = \frac{\pi^2}{6}$ and $\sum_{m>n\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^2} = \sum_{n>m\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^2} = \frac{\pi^4}{120}$ we now have

$$\zeta(4) = \left(\frac{\pi^2}{6}\right)^2 - 2\left(\frac{\pi^4}{120}\right) = \frac{\pi^4}{90}$$

□

Proposition 10. *The value of $\zeta(6)$ is*

$$\zeta(6) = \frac{\pi^6}{945}$$

Proof. Consider the expansion of $(\sum \frac{1}{m^2})(\sum \frac{1}{n^4})$. It is equivalent to the cases of $(n > m) \cup (m > n) \cup (m = n)$ so we have

$$\zeta(2)\zeta(4) = \sum_{n>m\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^4} + \sum_{m>n\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^4} + \sum_{n\geq 1}^{\infty} \frac{1}{m^6}$$

Since $\zeta(6) = \sum_{n\geq 1}^{\infty} \frac{1}{m^6}$, we can rearrange the equation so that

$$\sum_{n>m\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^4} + \sum_{m>n\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^4} = \zeta(2)\zeta(4) - \zeta(6) \tag{3}$$

Now we consider (2) again and compare the x^6 terms and multiply both sides by

$$\frac{\pi^6}{7!} = \sum_{k>m>n\geq 1}^{\infty} \frac{1}{k^2}\frac{1}{m^2}\frac{1}{n^2} \tag{4}$$

Now consider the expansion of $\zeta(2)^3$.

$$\zeta(2)^3 = 6\sum_{k>m>n}^{\infty} \frac{1}{k^2}\frac{1}{m^2}\frac{1}{n^2} + 3\sum_{k=n\neq m\geq 1}^{\infty} \frac{1}{k^2}\frac{1}{m^2}\frac{1}{n^2} + \sum_{k=m=n\geq 1}^{\infty} \frac{1}{k^2}\frac{1}{m^2}\frac{1}{n^2}$$

When all three terms are different, the sum is $\sum_{k>m>n}^{\infty} \frac{1}{k^2}\frac{1}{m^2}\frac{1}{n^2}$ and since there are 6 different ways of arranging the terms, we multiply it by 6. Then there are 3 possible ways of arranging the terms so that two terms are equal and one is different and 1 way where all three terms are equal.

We can see that the case when $k = m \neq n$ is the union of the cases when $k = m > n$ and when $k = m < n$. So using (3) we have

$$\sum_{k=n\neq m}^{\infty} \frac{1}{k^2}\frac{1}{m^2}\frac{1}{n^2} = \sum_{n>m\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^4} + \sum_{m>n\geq 1}^{\infty} \frac{1}{m^2}\frac{1}{n^4} = \zeta(2)\zeta(4) - \zeta(6)$$

Using (4) and the fact that $\sum_{k=m=n\geq 1}^{\infty} \frac{1}{k^2}\frac{1}{m^2}\frac{1}{n^2} = \zeta(6)$ we now have

$$\zeta(2)^3 = \frac{6\pi^6}{7!} + 3(\zeta(2)\zeta(4) - \zeta(6)) + \zeta(6)$$

Rearrange to get the final result

$$\zeta(6) = \frac{\frac{6\pi^6}{5040} + 3\zeta(2)\zeta(4) - \zeta(2)^3}{2} = \frac{\pi^6}{945}$$

□

Deriving a General Formula for $\zeta(2n),\ n \in \mathbb{Z}_{>0}$ using Bernoulli Numbers

In this section we generalise the computation for values of the ζ function at positive, even integers through a general formula using the Bernoulli Numbers[8].

Definition 11. *We define Bernoulli Numbers, B_k, using coefficients of the t^k-th term in the Taylor expansion of $\frac{t}{e^t-1}$.*

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$$

There is a recursive way to calculate the Bernoulli numbers[9]. First, we need a lemma.

Lemma 12. *The Taylor expansion of $\frac{e^t-1}{t}$ is*

$$1 + \frac{t}{2!} + \frac{t^2}{3!} + \dots$$

Proof. The Taylor expansion for e^t is

$$1 + t + \frac{t^2}{2!} + \dots$$

Subtract 1 from this and then divided by t to obtain our result. □

Theorem 13. *The first Bernoulli Number, B_0, is 1 and the others can be found with the recursive formula*

$$\binom{k+1}{k} B_k + \binom{k+1}{k-1} B_{k-1} + \binom{k+1}{k-2} B_{k-2} + \dots + \binom{k+1}{0} B_0 = 0$$

Proof. Using definition 11, we can construct the following equation

$$\left(\frac{e^t-1}{t}\right)\left(\sum_{k=0}^{\infty} \frac{B_k}{k!} t^k\right) = 1$$

Since any number multiplied by its reciprocal is 1. Using the Taylor expansion obtained in Lemma 12 and expanding the infinite sum on the left we can rewrite this as

$$\left(\frac{1}{1!} + \frac{t}{2!} + \frac{t^2}{3!} + \dots\right)\left(\frac{B_0}{0!} + \frac{B_1}{1!} t + \frac{B_2}{2!} t^2 + \dots\right) = 1$$

Comparing the coefficients of both sides at t^0, we have $B_0 = 1$. We can compare coefficients at the term t^k. The right hand side has a coefficient of 0 at all powers of t, so the expansion of the left hand side must also equal 0. So we have

$$\frac{1}{1!}\frac{B_k}{k!} + \frac{1}{2!}\frac{B_{k-1}}{(k-1)!} + \frac{1}{3!}\frac{B_{k-2}}{(k-2)!} + \dots + \frac{1}{(k+1)!}\frac{B_0}{0!} = 0$$

Note that if we multiply both sides by $(k+1)!$ the terms on the left hand side would have the form of the combination formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Simplifying, we have

$$\binom{k+1}{k} B_k + \binom{k+1}{k-1} B_{k-1} + \binom{k+1}{k-2} B_{k-2} + \dots + \binom{k+1}{0} B_0 = 0$$

In this way, we can find any B_k where k is a positive integer by substituting it into the formula, provided that we know the value of each of the previous Bernoulli Numbers. □

Our approach to proving the general identity will be similar to that of Proposition 3. First, we need the following.

Lemma 14. *The cotangent function can be written as*

$$\cot(x) = i + \frac{1}{x}\frac{2ix}{e^{2ix}-1}$$

Proof. We use the identity $\cot(x) = \frac{\cos(x)}{\sin(x)}$. From Euler's Formula, we know that $e^{ix} = \cos(x) + i\sin(x)$ and that $e^{-ix} = \cos(x) - i\sin(x)$. Solving this pair of simultaneous equations gives identities for sine and cosine in terms of e and i. So we can now write the cotangent as

$$\cot(x) = i\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = i\frac{e^{2ix}+1}{e^{2ix}-1}$$

By considering the numerator as $(e^{2ix}-1)+2$, we can further simplify our expression as

$$i\left(1 + \frac{2}{e^{2ix}-1}\right)$$

Expanding the bracket and multiplying both sides of the fraction by x gives

$$\cot(x) = i + \frac{1}{x}\frac{2ix}{e^{2ix}-1}$$

□

We are ready to give a classical proof of the formula for $\zeta(2k),\ k \geq 1$.

Theorem 15. *The general formula for the zeroes of the ζ-function at positive even integers is*

$$\zeta(2k) = \frac{(-1)^{k+1} 2^{2k-1} \pi^{2k} B_{2k}}{(2k)!} \tag{5}$$

Proof. We begin with an approach similar to that of Proposition 3, but we stop at the point

$$\cot(x) - \frac{1}{x} = \sum_{k=1}^{\infty} \left(-\frac{2x}{k^2\pi^2}\right)\left(\frac{1}{1-\frac{x^2}{k^2\pi^2}}\right)$$

By substituting $q = \frac{x^2}{k^2\pi^2}$ into the identity

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + \dots$$

we can rewrite our equation as

$$\cot(x) = \frac{1}{x} - 2\sum_{k=1}^{\infty}\left(\frac{x}{k^2\pi^2}\right)\left(1 + \frac{x^2}{k^2\pi^2} + \frac{x^4}{k^4\pi^4} + \dots\right)$$

Consider the infinite sum on the right-hand side, note that since $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$, when we expand the sum and collect all the $\frac{1}{k^n}$ terms together we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{k^2\pi^2}x + \frac{1}{k^4\pi^4}x^3 + \frac{1}{k^6\pi^6}x^5 + \dots\right)$$

$$= \frac{\zeta(2)}{\pi^2}x + \frac{\zeta(4)}{\pi^4}x^3 + \frac{\zeta(6)}{\pi^6}x^5 + \dots = \sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}x^{2k-1}$$

Applying Lemma 14 to the right-hand side then obtain the equation

$$i + \frac{1}{x}\frac{2ix}{e^{2ix}-1} = \frac{1}{x} - 2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}x^{2k-1}$$

Note that $\frac{2ix}{e^{2ix}-1}$ is in the form $\frac{t}{e^t-1}$ so, by Definition 11, we have

$$i + \frac{1}{x}\sum_{k=0}^{\infty}\frac{B_k}{k!}(2ix)^k = \frac{1}{x} - 2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}x^{2k-1}$$

Now multiply both sides by x and subtract 1. Note that -1 cancels out with the $k = 0$ term of the infinite sum on the left-hand side and ix cancels out with the $k = 1$ term. So we now have

$$\sum_{k=2}^{\infty}\frac{B_k}{k!}(2ix)^k = -2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}x^{2k}$$

For any k, compare the x^{2k} term on both sides to obtain

$$\frac{B_{2k}}{2k!}(2i)^{2k} = -2\frac{\zeta(2k)}{\pi^{2k}}$$

Note that $(2i)^{2k} = 2^{2k}(-1)^k$ and rearrange in terms of $\zeta(2k)$ to get the final result. □

A Proof of the General Formula Using Mathematical Induction and Newton's Identity

In this section we prove the general formula (5) for values of the zeta function via mathematical induction. This gives a more intuitive and elementary proof of the statement.

To begin our proof, note that we have

$$\zeta(2k) = (-1)^k\pi^{2k}\sum_{i=1}^{\infty}\left(-\frac{1}{i^2\pi^2}\right)^k \tag{6}$$

and by comparing the coefficients at x^{2k+1} between the product formula and the Taylor Expansion of the sine function we have

$$\sum_{1\le i_1<i_2<\dots<i_k<n}^{n}\left(-\frac{1}{i_1^2\pi^2}\right)\left(-\frac{1}{i_2^2\pi^2}\right)\cdots\left(-\frac{1}{i_k^2\pi^2}\right) = \frac{(-1)^k}{(2k+1)!} \tag{7}$$

Notice that the right hand side of (6) is a power sum and the left hand side of (7) is an elementary symmetric polynomial. So we can connect them using Newton's Identities [10].

Suppose that we have variables $x_1, \dots, x_n$. Then we define two types of symmetric polynomials.

Definition 16. *We define $p_k(x_1,...,x_n)$ as the k-th power sum so that*

$$p_k(x_1,...x_n) = \sum_{i=1}^{n} x_i^k$$

Definition 17. *We define $e_k(x_1,...,x_n)$ as the elementary symmetric polynomial such that $e_0(x_1,...,x_n) = 1$, $e_1(x_1,...,x_n) = \sum_{i=1}^{n} x_i$, $e_2(x_1,...,x_n) = \sum_{1\le i<j\le n}^{n} x_i x_j$ and, in general,*

$$e_k(x_1,...,x_n) = \sum_{1\le i_1<i_2<\dots<i_k<n}^{n} x_{i_1}\dots x_{i_k}$$

Theorem 18. *Newton's Identity states that*

$$\sum_{i=1}^{k}(-1)^{i+1}e_i p_{k+1-i} = p_{k+1} + (-1)^{k+1}e_{k+1}(k+1)$$

Proof. See reference [10]. □

Note that Newton's identities still make sense if the number of variables is infinite, i.e the variables are $x_1, x_2, \dots, x_n, \dots$, so we can begin our proof. The base of the proof when $k = 1$ is obvious and the induction step could be simplified to an identity.

Theorem 19. *The induction step in the proof of formula* (5) *is equivalent to the identity*

$$\sum_{i=1}^{k+1}\binom{2k+3}{2i}2^{2i-1}B_{2i} = k+1$$

Proof. We set up our p_k and e_k with $x_1,...,x_n$ such that $x_i = -\frac{1}{i^2\pi^2}$. Note the relationship we created p_k and the $zeta$-function as well as e_k and the sine function.

$$p_k = \sum_{i=1}^{\infty}\left(-\frac{1}{i^2\pi^2}\right)^k = \frac{(-1)^k}{\pi^{2k}}\zeta(2k)$$

$$e_k = \frac{(-1)^k}{(2k+1)!}$$

From Theorem 18 we have

$$\sum_{i=1}^{k}(-1)^{i+1}e_i p_{k+1-i} = p_{k+1} + (-1)^{k+1}e_{k+1}(k+1)$$

Substituting e_k and p_k using our previous identities we have

$$\sum_{i=1}^{k}(-1)^{i+1}\frac{\zeta(2i)}{(2k-2i+3)!\pi^{2i}} = \frac{(-1)^{k+1}}{\pi^{2k+2}}\zeta(2k+2) + \frac{k+1}{(2k+3)!}$$

Assuming that the formula works for $n = 1,...,k$, it works for $n = k+1$ if and only if when we substitute all the formulas into the identity. Further substituting using our general formula for the ζ-functions to obtain

$$\sum_{i=1}^{k}\frac{2^{2i-1}B_{2i}}{(2i)!(2k-2i+3)!} = -\frac{2^{2k+1}B_{2k+2}}{(2k+2)!} + \frac{k+1}{(2k+3)!}$$

Where we can simplify the (-1) terms because on the left hand side $(-1)^{2k+2}$ will always be positive while on the right hand side $(-1)^{2k+1}$ is always negative. Note that on the right hand side, $\frac{2^{2k+1}B_{2k+2}}{(2k+2)!}$ is in the same format as the sum and is the case where $i = k+1$ so we can move it to the left and expand the sum to $k+1$. Then we multiply both sides by $(2k+3)!$ and note that on the left we have $\frac{(2k+3)!}{(2i)!(2k+3-2i)!}$ which resembles the combination formula $\binom{2k+3}{2i}$ so we can simplify our result to

$$\sum_{i=1}^{k+1}\binom{2k+3}{2i}2^{2i-1}B_{2i} = k+1$$

□

Now, all that remains to prove (5) is showing that our identity holds.

Theorem 20. *Formula* (5) *holds.*

$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}B_{2k}}{(2k)!}$$

Proof. The identity in Theorem 19 is obtained by assuming the validity of the general formula. Therefore if we can prove this identity we have thus proved the general formula. To do this we adopt a similar approach to Theorem 13. We start by considering the equation

$$\frac{e^t-1}{t}\frac{2t}{e^{2t}-1} = \frac{2}{e^t+1} \quad (8)$$

On the left hand side we can write $\frac{e^t-1}{t}$ in terms of Taylor Expansion and $\frac{2t}{e^{2t}-1}$ in terms of Bernoulli numbers. So

$$\sum_{k=0}^{\infty}\frac{t^k}{(k+1)!}\sum_{k=0}^{\infty}\frac{B_k}{k!}2^k t^k = \frac{2}{e^t+1}$$

We shall now prove that on the right hand side the coefficients at even powers of t are equal to 0. Let $f(t) = \frac{2}{e^t+1}$ and note that

$$f(t) + f(-t) = \frac{2}{e^t+1} + \frac{2}{e^{-t}+1} = 2 \quad (9)$$

Now, consider their Taylor expansions. We have

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots$$
$$f(-t) = a_0 - a_1 t + a_2 t^2 - \ldots.$$

so that

$$f(t) + f(-t) = 2a_0 + 2a_2t^2 + 2a_4t^4 + \ldots$$

Now equation (9) implies that $a_0 = 1$ and coefficient at all other even powers equal 0.

Now we can compare the coefficients at t^{2n+2} of Equation (8). So we have

$$\frac{B_{2n+2}}{(2n+2)!}2^{2n+2} + \frac{B_{2n+1}}{(2n+1)!2!}2^{2n+1} + \frac{B_{2n}}{2n!3!}2^n + \cdots = 0$$

Similar to Theorem 13, we multiply both sides by $(2n+3)!$ to obtain

$$\binom{2k+3}{2n+2}B_{2n+2}2^{2n+2} + \binom{2k+3}{2n+1}B_{2n+1}2^{2n+1} + \binom{2k+3}{2n}B_{2n}2^{2n} + \cdots = 0 \quad (10)$$

from Theorem 13 and all other odd Bernoulli numbers equal 0. So we can simplify our expression, writing only in terms of even Bernoulli numbers, B_0 and B_1.

$$\sum_{i=1}^{k+1}\binom{2k+3}{2i}2^{2i}B_{2i} + \binom{2k+3}{1}2B_1 + \binom{2k+3}{0}B_0 = 0$$

Simplifying this we have

$$\sum_{i=1}^{k+1}\binom{2k+3}{2i}2^{2i}B_{2i} - (2k+3) + 1 = 0$$

We then move $2k+3$ and 1 to the right hand side and then divide both sides by 2 to obtain our induction step

$$\sum_{i=1}^{k+1}\binom{2k+3}{2i}2^{2i-1}B_{2i} = k+1$$

□

Conclusion

In this paper we summarized some key findings regarding the Riemann Zeta Function. We began with a proof of the product formula which seemed unrelated to our topic but turned out to be key in finding the value at small numbers. After showing a few case-by-case examples of evaluating the Zeta Function at lower values, we extended the result to include all positive even integer values by showing the derivation of the general formula. Our new proof using mathematical induction verifies the validity of this general formula. However, we note that due to the method we used we are only able to evaluate the Zeta Function in some particular cases. Its value at odd integers, fractions, as well as negative values must be found using another method. Most importantly, results regarding the Riemann Zeta Function at complex values (the Riemann Hypothesis) remains to be discovered, and has proven to be one of the most important and fascinating mysteries to be uncovered in modern mathematics.

References

1 Wikipedia. Basel Problem (2024) https://en.wikipedia.org/wiki/Basel_problem

2 E.Bombieri. The Riemann Hypothesis - official problem description (2000) https://web.archive.org/web/20151222090027/http://www.claymath.org/sites/default/files/official_problem_description.pdf

3 ProofWiki. Riemann Zeta Function at Non-Positive Integers (2024) https://proofwiki.org/wiki/Riemann_Zeta_Function_at_Non-Positive_Integers

4 G.H.Hardy, J.E.Littlewood. The zeros of Riemann's Zeta-Function on the critical line. (1921) https://zenodo.org/records/1447415

5 K.Alladi, C.Defant. Revisiting the Riemann Zeta function at positive even integers (2018) https://people.clas.ufl.edu/alladik/files/ReprintIJNTzeta2k.pdf

6 A.Banner. *The Calculus Lifesaver: All the Tools You Need To Excel at Calculus* (2007) https://books.google.co.uk/books?id=OrumDwAAQBAJ&redir_esc=y

7 NatLab. Product Formula for the Sine Function (2023) https://ncatlab.org/nlab/show/product+formula+for+the+sine+function

8 Mathematics Stack Exchange. Ways to prove Euler's formula for $\zeta(2n)$ (2015) https://math.stackexchange.com/questions/1322604/ways-to-prove-eulers-formula-for-zeta2n

9 E.Weissenstein. Wolfram Mathworld. Bernoulli Number https://mathworld.wolfram.com/BernoulliNumber.html

10 M.Mosse. Newton's Identities (2019) https://web.stanford.edu/~marykw/classes/CS250_W19/Netwons_Identities.pdf

