Field Theories in Inflationary Cosmology

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The following work will be exploring the prevailing frameworks for the modeling of the early universe's inflation; namely old inflation, slow-roll inflation, and ultra slow-roll inflation given a quadratic field potential. We then go on to explore the various assumptions of the models in terms of field dynamics, cosmological parameters, and thermodynamic criteria. A review of the shortcomings of these models is undertaken; followed by the proposal of two research pathways in order to address the same: galactic redshift analysis and the spectral analysis of the cosmic microwave background. These research pathways will be investigated in the future in coming endeavors: utilizing perturbation theory and early universe cosmology to match theoretical CMB patterns with current CMB patterns; and the corroboration of universe age in e–folds from theory with the heuristically determined age of the cosmos.

1 Introduction

1.1 Modern Cosmology

As a preface to the research and literature review conducted hereafter, it is valuable to provide a general overview of modern cosmology as well as its most prevalent challenges. This will provide background for the mathematical exploration of the current variations in the inflationary models in the field of cosmology.

1.1.1 The Contemporary Model

Cosmology concerns itself with the origin and evolution of the universe. In the decades leading up to the development of inflation theory – in which the Lambda–Cold Dark Matter model is the most recent[1] – researchers faced two primary conceptual conflicts, outlined below.

1.1.2 The Flatness Problem

First, the flatness problem refers to the substantially unlikely state of the current universe wherein the curvature of spacetime is incredibly fine-tuned. It is at a state of critical density so well-balanced that uncertainty has been cast on the nature of this precarious equilibrium. There is widespread doubt as to how natural the model we have created is if such fine-tuning is required to make it heuristically accurate.

To clarify the specifics of this challenge, we can undertake a mathematical exploration of this problem. The mathematics explored below are a summary of the induction in Prof. Alan Guth's famous 1981 paper that founded the field of inflation[2].

To begin, we start with the Friedmann-Lemaître-Robertson-Walker metric [Equation 1].

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right](1),$$

describing the path between two points in positively curved qascetime [$k = +1$], negatively curved gpacetimes [$k = -1$], or flat pacetime [$k = 0$], where k is a pyitleas parameter that has one of three values based on the curvature of the universe as described above. Equation 1 is the polar coordinate form of the metric[3]. d^2 is the distance between two points in space, while r, k, θ, and Φ are all parameters that are based on the space the two points are within

The second postulate of the cosmological model is that the universe expands in apacetime according to the at any time t. We can deacribe the evolution of $a(t)$ through the following Einstein equations [Equations 2 and 3]

$$\ddot{R} = -\frac{4\pi}{3}G(\rho + 3p)R \qquad (2)$$

$$H^2 + \frac{k}{R^2} = \frac{8\pi}{3}G\rho \qquad (3)$$

Both of the above equations can berderived by plugging in the parameters of the field's equation of state into Equation 1.

The third parameter we introduce[2] is that energy is conserved according to Equation 4.

$$\frac{d}{dt}\left(\rho R^3\right) = -p\frac{d}{dt}\left(R^3\right) \qquad (4)$$

Equation 4 implies that the change of the energy density of the universe as it expands is equal to the change in the size of the universe multiplied by the negative pressure of the universe. This wallalsobeseglored further in Section 2 , where the mechanism itself will be explored in further detail.

Following from the above, assuming the adiahaticity - or the lack of overall change in entropy - of the universe, we get Equation 5.

$$\frac{d}{dt}\left(sR^3\right)=0 \qquad (5)$$

Qualitatively, a constant entropy means that its derivative with respect to time must be 0; where is the entropy density. While this is certainly disproven in modern cosmology and physics, it is important to note that this was a prevailing assumption in Guth's time and work.

Therefore, along the same lines, it is necessary that we define the prevailing thermodynamic norms and statutes in the 1980s before working with the development of entropy for old inflation.

Models in the late 20th Century assumed a gas of particles with bosonic and fermionic degrees of freedom and respectively, the functions of density in kilograms per cubic meter, entropy in joules per Kelvin, and particle number density per cubic meter are as follows [Equations 6, 7, 8] keeping in mind equations 9 and 10[4].

$$\rho=3p=\frac{\pi^2}{30}\mathfrak{N}(T)T^4 \qquad (6)$$

$$s=\frac{2\pi^2}{45}\mathfrak{N}(T)T^3 \qquad (7)$$

$$n=\frac{\zeta(3)}{\pi^2}\mathfrak{N}'(T)T^3 \qquad (8)$$

In which:

$$\mathfrak{N}(T)=N_b(T)+\frac{7}{8}N_f(T) \qquad (9)$$

$$\mathfrak{N}'(T)=N_b(T)+\frac{3}{4}N_f(T) \qquad (10)$$

Equations 9 and 10 are partition functions, which calculate the fermionic and bosonic degrees of freedom respectively, and ase calculated by the product of the species, particle-antiparticle pairs, and spin states. These values essentially quantify the contributions that fermionic particles and bosonic particles have to the density of the system. In Fquation 8, $\zeta(3)$ denotes the Riemann Zeta function with an argument of 3 , approximately equal to 1.202 - which is a value known as A Avedisconstant, an important number in themodynamics.

Recall equation 3 , utilized to describe the evolution of the scale factor of the universe:

$$H^2+\frac{k}{R^2}=\frac{8\pi}{3}G\rho \qquad (11)$$

Rewritten in terms of temperature T in Kelvin, equation 12 is.sbtained:

$$\left(\frac{\dot{T}}{T}\right)+\varepsilon(T)T^{\alpha}=\frac{4\pi^2}{45}G\mathfrak{N}(T)T^4 \qquad (12)$$

Here, H , or the Hubble constant, is another notation for the scale factor of the universe. As the change in temperature is proportional to the scaling of the universe, it is a good approximation for the Hubble factor. The dot denotes differentiation with respect to time. The second term $\varepsilon(T)$ is a function of temperature, radius, and curvature, defined as Equation 13 . It makes corrections to the rate of change of temperature based on the curvature of the universe.

$$\varepsilon(T)=\frac{k}{R^2T^2}=k\left[\frac{2\pi^2}{45}\frac{\mathfrak{N}(T)}{S}\right]^{\frac{2}{3}} \qquad (13)$$

The value of ρ is plugged in to arrive at Equation 13. It is a matter of notation. Equation 13 is a logical consequence of the relation $S=R^3s$ where S is the total entropy in a volume with the radius of curvature R . We are one to plug in $\varepsilon(T)$ back into Equation 3, one would find Equation 13.

Next, by calculating the photon, electron, and neutrino contribution to the entropy given by $\mathfrak{R}(T)$ and $\mathfrak{R}'(T)$, we anrive at the conclusion that entropy is greater than 10^{86} Joules per Kelvin, which is an incredibly high amount.

$$S>10^{86}$$

Plugging S back into Equation 13,

$$|\varepsilon|<10^{-58}\mathfrak{Y}^{\frac{2}{3}}$$

Now, using the investigated values, if we compare the difference between the current energy density and the critical energy density with the current energy density, we get Equation 16:

i.e, the universe's present density matches the critical density of the universe [the borderline between a flat universe and positively / negatively curved universe] with an error margin of less than 3×10^{-58}.

Logically, if

$$\begin{gathered}\rho_{\text{current}}\simeq\rho_{\text{crit}}\\ \Rightarrow\rho_{\text{current}}-\rho_{\text{crit}}\simeq0\\ \text{or }\rho_{\text{current}}-\rho_{\text{crit}}\ll1\\ \therefore\frac{\rho_{\text{current}}-\rho_{\text{crit}}}{\rho_{\text{current}}}\ll1,\text{ or }\frac{\rho_{\text{current}}-\rho_{\text{crit}}}{\rho_{\text{current}}}\simeq0\end{gathered}$$

Since the error margin is a factor of 3×10^{-58}, it is clear that the current energy density of the universe and the critical energy density are virtually identical.

This aforementioned fine-tuning isbyorm as the flatness problem[2].

1.1.3 The Horizon Problem

The second challenge in the field of cosmology is known as the horizon problem. Extrapolating from the above values will allow us to undertake the task of achieving a mathematical description of the same. Once again, the deduction below is a summary of the description of the horizon problem in Prof. Guth's 1981 paper on inflationary theory[2].

Returning to Equation 11, we solve for temperatures above mass thresholds through certain mathematical transformations.

$$\left(\frac{\dot{T}}{T}\right)+\varepsilon(T)T^{2}=\frac{4\pi^{2}}{45}G\mathfrak{N}(T)T^{4} \qquad (18)$$

By ignoring the $|\varepsilon|_{\text{term, we get}}$

$$\Rightarrow T^{2}=\frac{M_{P}}{2\gamma t} \qquad (19)$$

Where γ is a constant used to relate the partition function, Equation 9, and temperature in Equation 19:

$$\gamma^{2}=\frac{4\pi^{3}}{45}\mathfrak{N}$$

Next, it is important to note that if entropy is to be conserved, the term RT must be constant, as the distribution of the temperature over a space of radius with curvature R must be constant. Following from this, we can derive Equation 20:

$$R\propto\frac{1}{T} \qquad (20)$$

This concludes the discussion of the characteristics and prerequisites of the cosmological model needed for this section of the paper.

The next component we need to explain the horizon problem quantitatively is the distance a light ray has traveled by a time t. We can use the formula for the causal horizon distance with the following integral as seen in Equation 21.

$$l(t)=\int_{0}^{t}dt'R^{-1}\left(t'\right)=2t \qquad (21)$$

This integral derives the distance light can travel, in meters, from a certain point in time as the universe evolves.

To apply it to our universe, we can derive the causal horizon we currently observe by assuming the conservation of entropy. If entropy is constant, we can find the change in distribution of the entropy over the radius, in meters, of the observable universe as seen in equation 22.

$$L(t)=\left[s_{p}/s(t)\right]^{\frac{1}{3}}L_{p} \qquad (22)$$

L_p is the present radius of the observable universe [i.e. 10^{10} ly where ly is light years, or the distance travelled by light in one year, which is equivalent to $9.5*10^{15}$ meters] and s_p denotes the present entropy density.

Qualitatively, the previous and next steps find the area over which events are causally connected - i.e. can affect each other in real time.

If an event in spacetime is further than the distance light can travel from you at that moment, it is causally, disconnected from you, as the speed of light is the speed limit of the universe.

Now that both values have been defined in equations 21 and 22, we can compare them to explain the horizon problem. By finding the ratio of the two volumes l^3 and L^3, we find that

$$\frac{l^{3}}{L^{3}}=\frac{11}{43}\left(\frac{45}{4\pi^{3}}\right)^{\frac{3}{2}}\mathfrak{N}^{-\frac{1}{2}}\left(\frac{M_{P}}{L_{p}T\gamma T}\right)=4\times10^{-89}\mathfrak{N}^{-\frac{1}{2}}\left(\frac{M_{P}}{T}\right)^{3} \qquad (23)$$

with $\mathfrak{N}\approx10^{2}$ and $T_0\approx10^{17}$GeV, we find that

$$\frac{l^{3}}{L^{3}}=10^{-83} \qquad (24)$$

implying that the region of our observed universe is approximately 83 orders of magnitude greater in size than the causal, or physical, horizon distance.

This seems illogical as disconnected causal regions would then fail to explain the large-scale homogeneity and isotropy of the universe. There is no reason for the disconnected patches to evolve identically.

This is known as the horizon problem[2].

1.2 Quantum Field Theory

In order to set the stage for the discussion of more contemporary models, it is important to lay down the groundwork of quantum field theory, otherwise known as QFT.

In essence, quantum field theory postulates the existence of a set of fields that permeate spacetime; each belonging to a particle. Fields are mathematical constructs that assign a scalar or vector to all points in spacetime. Excitations in fields are construed as the manifestation of their respective particles[5].

These excitations are the wave functions of the particles; i.e. probability distributions of the location and momentum of a particle. These waves are manifest in the fields themselves as fluctuations; which are then directly quantised rather than the particles themselves.

The interactions of these waves, or particles, can be described mathematically, or through a visual representation – i.e. Fenyman diagrams, in which the paths of particles are plotted on graphs against time.

Overall, quantum field theory unifies various theories, including quantum electrodynamics, quantum chromodynamics, the standard model of particle physics, and electroweak theory.

1.3 Relativity

While quantum field theory deals with the smallest scales of the universe, we must also consider its converse: general relativity, describing the large-scale structure and curvature of the universe.

The brainchild of the infamous Einstein, relativity postulates the tie-up of space and time into a unified manifold monikered spacetime, whose shape is decided by the mass and energy content of the universe.

The resulting geometry of spacetime is what we experience as gravity, i.e. bodies of mass and light traveling on a straight line but in curved spacetime being perceived as gravitational attraction.

General relativity's counterpart, special relativity, is the second component of relativity. Special relativity postulates that time is not absolute; and that the speed of light in a vacuum is the cosmic speed limit. Furthermore, time dilates around big masses, and length contracts at high speeds. Einstein's notorious equation is an essential component of this framework.

1.4 Effective Field Theories

Effective field theories are not a singular concept, but an approach to describing our universe. EFTs are special in that they are specialized for certain scales; they work on the key assumption that our universe can be explained through different frameworks at different sizes.

A prime example of this is quantum field theory and general relativity - both of which only function at their respective scales. Unifying the various exclusively functioning theories currently used to describe the universe is one of the greatest challenges of modern physics.

There is a wide array of issues, both conceptual as well as intrinsic, to be discussed within unification; ranging from the occurrence of infinities, which are not present in nature; the conceptual conflict of frameworks in various ases, including dark matter, or the nature of gravity; as well as the occurrence of questions about existence, teleology, and philosophy.

1.5 Methods

The search strategy incorporated in this research was to use primary search engines such as Google to perform a preliminary search for solely articles, recorded lectures, or university talks that were relevant to the search. If no relevant sources were found, databases such as ResearchGate and Google Scholar were used to find a wider variety of papers in terms of date of publication as well as authors and regions.

Keywords were limited solely to the concept and the medium of the information. An example search would be 'perturbation theory cmb stanford.' Unless absolutely necessary, no websites were used. The papers, talks, and lectures used were vetted on the following factors: a recent date of publication [unless it was for a review of early theory], reputable institutions of publication, such as universities or well–known journals, and quality of language.

Once a source was found, it was read through thoroughly to find only relevant information. The latter could be in the form of an equation, a quote, or an explanation of an unfamiliar phenomenon. To structure its reference, the source's authors were searched for at the top of the page, the bottom of the page, or on the main page of the website being used.

For papers in which the PDF was given or the DOI was given, the PDF was opened and information was presented on the front page, or for the DOI, information was stored in their main database and was extracted using a bibliography tool monikered NoodleTools.

To synthesize my information, I had first organized a flow of narrative. I planned and followed a set of bullet points to ensure that my sections flow into each other languidly. I also ensured the implementation of conclusions and introductions between sections to facilitate the transitions.

2 Literature Review

2.1 Modern Theories

2.1.1 Old Inflation

Guth's proposed theory – now dubbed old inflation – was preceded only by a simplistic framework for the description of the cosmos: a homogeneous, isotropic expanding universe[2]. This theory, however, carried certain irrefutable scientific and mathematical barriers, as mentioned in Sections 1.1.2 and 1.1.3. The same is summarized below from Guth's paper on the same.

To address the same barriers, Guth, in essence, formulated a model in which he postulates a period of rapid expansion in the early universe monikered inflation. His paper outlines the potential resolution of the flatness and horizon problems through inflation, the mechanisms that drive inflation, and evident shortcomings of the theory.

To begin with, Guth outlines the discrepancy of rampant causal disconnection in a homogeneous universe [i.e. the horizon problem] as well as the irregularly precise fine-tuning of λ_M [the matter (baryonic + dark) density of the universe] to specific values stringently required for the flatness of the universe [i.e. the flatness problem].

Guth introduces a scalar field ϕ known as the inflaton field that drives the expansion of the universe as a function of its potential energy.

The nascent universe has a sea of – what can be considered – massless particles in an adiabatic system heated to extremely high temperatures. Analogously, the potential density of the field $V(\phi)$ is at its maximum. Consequently, spacetime begins to stretch out exponentially - while also increasing the distribution of matter and radiation across the universe - subsequently decreasing $V(\phi)$. The expansion of spacetime as a result of the inflaton field is caused by its negative pressure that pushes outwards on the universe: creating a gravitationally repulsive force reminiscent of Einstein's cosmological constant.

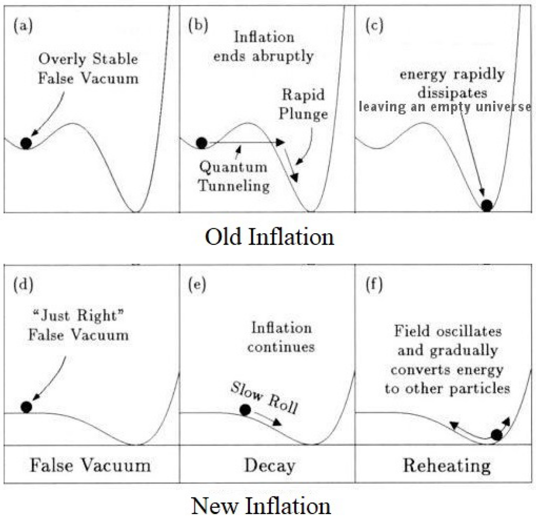

Fig. 1 Scalar Field Dynamics [6]

As $V(\varphi)$ decreases, the inflation field approaches its zero-point energy, at which point inflation would "stop." However, while the inflaton field harbors a definite zero-point energy, as do all scalar fields, its potential curve also traces a local minimum known as the false vacuum.

This is a key characteristic of the old inflation theory. As seen in Fig. 1, the field's potential can decay from its local maximum into the overly stable false vacuum. Herein, the universe supercools through expansion as the inflaton field is virtually "stuck" in the valley of the local minimum. As a result, the universe continues to expand rapidly and - most importantly - indefinitely.

Old inflation and the phase transition of the inflaton field provided an adequate explanation for the flatness problem in that any incident curvature of the universe at its 'birth' is sufficiently stretched out as to have it appear locally flat: i.e. in de Sitter space.

In addition to the novel mechanism, Guth also proposes the abandonment of the aforementioned theory of an adiabatic universe. With a dynamic entropy S that does not obey the conventional relationship of a total entropy $S = R^3$ where R is the radius of curvature and s is the entropy density, the present total entropy S_p and initial total entropy are now related by equation 25.

$$S_p = Z^3 S_0 \qquad (25)$$

Z is considered a "large factor.' Mathematically, the entropy of the universe is a key factor in the calculation of the radius, in meters, of the observable universe L_p, where

$$L(t) = \left(\frac{S_p}{s(t)}\right)^{1/3} L_p \qquad (26)$$

With the introduction of a factor Z, the right hand side of Equation 26 is significantly diminished - i.e. the radius of the observable universe is highly decreased, therefore limiting it to a value within the horizon at the relevant time. This means the evolution of the universe - at all points in spacetime - remains within the causal horizon; therefore possibly eradicating the horizon problem. Z being greater than a value of is the only constraint for this model [2].

2.1.2 Slow Roll Inflation

While old inflation certainly was an innovative new framework, it harbored obvious inconsistencies. Over a span of decades, various modifications were placed on old inflation in order to create 1. More logical transitions from the false vacuum to the true vacuum, 2. Consistencies with observations, and 3. Consistency with the estimated age of the universe.

The dominant model that emerged from these corrections is referred to as slow-roll inflation. Slow-roll inflation is a modern modification of old inflation. To discuss the mechanics of the field, we must first understand the application of this scalar field to cosmology.

We know that the field exerts a repulsive gravitational force on its surroundings. This is, in fact, caused by the inherent negative pressure of the inflaton field, which will be discussed further in Section 2.1.3.

The initial conditions of the universe, formalized by the FLRW metric [then the Robertson-Walker metric] [Equation 1] and Einstein's equations [Equations 2 and 3], lead to the aforementioned challenges. However, when a transient period of exponential expansion is introduced just after the birth of the universe, both the horizon as well as the flatness problems are addressed, as seen below.

2.1.3 Horizon Problem Solution

To address the horizon problem, the slow-roll model postulates that the scalar inflation field exerts negative pressure, leading to a repulsive gravitational effect. To find this, we can reason the following:

The change in the energy density of the field can be modeled by placing it in the second Friedmann equation in terms of the density and equation of state [6].

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho - \frac{p}{c^2}\right) \qquad (27)$$

Since the energy density must stay constant when the field is stuck in the false vacuum, as explained above, Equation 27 implies

$$\dot{\rho} = 0 \rightarrow p = -\rho c^2 \qquad (28)$$

As when we substitute the same value for ρ into the equation above, we find a factor of 0 on the right hand side, therefore telling us that , or the change of the energy density with respect to time, is zero[7].

The second Friedmann equation tells us

$$\ddot{a} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2} \right) a \qquad (29)$$

As just derived above in Equation 28,

$$p = -\rho c^2 \qquad (30)$$

Plugging the same value for ρ into the second Friedmann equation would lead to:

$$\frac{4\pi}{3} G < 0 \text{ and } \left(\rho + \frac{3p}{c^2} \right) < 0 \qquad (31)$$

Rearranging for the scale factor $\frac{ddota}{a}$, or H,

$$\frac{\ddot{a}}{a} > 0 \qquad (32)$$

This means that there is an accelerating Hubble expansion rate, or an accelerating scale factor . An accelerated expansion is also observed experimentally by cosmologists[7].

Following from the above, during inflation, if we assume an expanding patch of space of size where the average value of the inflaton potential is approximately 0, or

$$\langle \phi \rangle = 0 \qquad (33)$$

We can model inflation starting with this patch. To find the next series of events, we can approximate it to be a homogeneous Robertson-Walker patch and utilize the first Friedmann equation that describes its evolution[6],

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho_f \qquad (34)$$

where ρ_f is the density of the inflaton field [assuming the patch is dominated by the inflation field].

Expressing the accelerating scale factor a as a function of time, a(t) , through certain mathematical transformations, one derives Equation 36.

$$a(t) = \text{const} e^{\chi t} \qquad (35)$$

where[7]

$$\chi = \sqrt{\frac{8\pi G}{3} \rho_f} \qquad (36)$$

This can be generalized to any universe with initial conditions where if the pressure is negative, then

$$p = -\rho c^2 \qquad (37)$$

$$\rho > 0 \qquad (38)$$

This is monikered the cosmological no-hair conjecture, i.e. that any system with an average inflation potential of 0 and negative pressure, as seen above, will evolve to locally resemble a flat exponentially expanding spacetime, i.e. de Sitter space[6]. This will be key in the solution of the horizon problem.

Our next step to solve the problem is to consider a distance

$$\triangle r(t_1, t_2) \qquad (39)$$

which denotes the coordinate distance that light travels from t_1 to t_2 . To find the value of the same, we integrate over the coordinate distance of light, which is given by

$$\therefore \Delta r(t_1, t_2) = \int_{t_1}^{t2} \frac{c}{a(t)} dt \qquad (40)$$

Substituting the value obtained for $a(t)$ above, we get Equation 42 .

$$\rightarrow \Delta r(t_1, t_2) = \int_{t_1}^{t_2} \frac{c}{a(t)} dt$$
$$= \frac{c}{\text{const } \chi} \left[e^{-\chi t_1} - e^{-\chi t_2} \right] \qquad (41)$$

Supposing the light ray travels for an arbitrarily long time,

$$t_2 \rightarrow \infty$$
$$\lim_{t_2 \rightarrow \infty} e^{-\chi t_2} = \lim_{t_2 \rightarrow \infty} \frac{1}{e^{t_2}} \simeq 0 \qquad (42)$$

This renders the integral solution for the coordinate light distance

$$= \frac{c}{\text{const } \chi} \left[e^{-\chi t_1} \right] \qquad (43)$$

This implies that there is a limit to the range of causality in the universe, which can be expressed as[7].

$$a(t) \lim_{t \rightarrow \infty} \Delta(t_1, t_2) = \text{ event horizon distance } = c\chi^{-1} \qquad (44)$$

Therefore, we can postulate that if anything emits a light ray at a distance of one Hubble length or longer from us, we will never receive it.

There is an inherent event horizon in de Sitter space. This means that once a sizable de Sitter region is created, it is essentially ‘protected’ from anything outside its event horizon; as not even light can enter it fast enough to counter the expansion of space.

The next step in the solution is to find the mass density of the inflaton field. Through dimensional analysis in grand unified theories of energy scales 10^{16} GeV, we find that

$$p_f \approx \frac{E_{GUT}^4}{c^5} \approx 2.3 \times 10^{81} \text{ g cm}^{-3} \quad (45)$$

This is a shockingly high mass density. Substituting the same for χ in the expansion rate [Equation 36]

$$\chi \approx 2.8 \times 10^{-38} \quad (46)$$

and

$$c\chi^{-1} \approx 8 \times 10^{-28} \text{ cm} \quad (47)$$

This means we start off with a patch of the universe of the size above, and ending at scales of , expanding until at the end of inflation[7].

Nearing the end of inflation, the potential oscillates at the zero-point energy of the field, effectively reheating the universe at scales of $10^{16} GeV$.

To find the size of this patch today, we can find the ratio of the temperature at the beginning of inflation to the temperature in the present, and multiply it by 10cm to get:

$$\frac{10^{16}\text{GeV}}{K_B 2.7K} \times 10 \text{ cm} = 450 \times 10^9 ly \quad (48)$$

This is a valid calculation as the scale factor is directly inversely proportional to the temperature of the universe. While this value is approximately 10 times larger than our current observations, it is permissible as we could also use a value of instead to make up for it.

The size of this horizon invalidates what is known as the horizon problem, in that the universe should technically be filled with causally disconnected patches that would not have been in contact with each other enough to create a homogeneous space. Here, we see a Hubble region consistent with our observations, essentially eradicating this concern[6].

2.1.4 Flatness Problem Solution

The second major challenge slow-roll inflation solves is the flatness problem. This is achieved by performing the following mathematics:

We know that the Friedmann equation states

$$H^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \quad (49)$$

As derived above in Equation 48, a grows by a factor of 10^{28} , effectively reducing the curvature term in the equation by a factor of 10^{56} as kc^2 is divided by the square of a .

Therefore, at smaller scales, the universe can be approximated as flat due to a negligibly low curvature – approximated as flat[7]. This essentially provides reasoning for the flatness problem; i.e. why the universe's density is so fine-tuned to the critical density; and analogously, the curvature of the universe being observably flat.

2.1.5 The Cosmic Microwave Background

Lastly, the homogeneity and isotropy of the universe can be explained using this model in the framework of quantum mechanics.

Gravitational instabilities play a key role in creating inhomogeneities. The aggregation of matter as gas clouds, stars, galaxies, galaxy clusters, superclusters, and eventually, filaments, creates inhomogeneities in the overall cosmic microwave background. These lead to the amplification of the fluctuations in the CMB.

The origin of the CMB ripples must yet be investigated, however.

Classically, inflation would lead to a uniform mass density. However, quantum mechanics predicts an almost uniform density after inflation. The effect of the quantum fluctuations can be calculated. This is further explored in Section 2.2.2.

2.1.6 Ultra–Slow Roll Inflation

Ultra slow-roll inflation, a more contemporary theory, is an extension of the more well-known slow-roll framework - in which, rather than being stuck in a false vacuum, the field rolls down an extremely flat potential curve and oscillates at the zero-point energy level before settling.

Ultra slow-roll inflation attempts to resolve the prediction that slow-roll inflation may stop at extremely flat – although not inflectional – points of the potential. This is due to the fact that slow roll creates a fall in the inflaton field's potential more rapidly than if the field were in free-fall. As free-fall is the limit of the speed of decrease of any quantity, this is a discrepancy in the model.

To solve this, the inflaton field is expected to transition from slow-roll to ultra slow-roll inflation at the flattest areas of the potential, particularly inflection points. Mathematically, this happens when the acceleration locks with the potential slope rather than the friction.

To understand this, we must first explore the large-scale dynamics of the inflation field.

Equation 50 gives the field's Lagrangian[8].

$$L = -\frac{1}{2}\eta_{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} - V(\phi) \quad (50)$$

Computing the stress-energy-momentum tensor for the field gives us a set of two equations for the energy density and the pressure of the field. We integrate the Lagrangian and scalar over 4 dimensions[9], or 3 dimensions of space and 1 dimension of time, giving us the action

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \quad (51)$$

By adding the Ricci tensor action, which is a calculation that quantifies the curvature of space in the universe, and the field action, which is Equation 52, one can compute the stress-energy-momentum tensor: which describes the density and distribution of energy in spacetime. The exact calculation is outside the scope of this research, however. Using the stress–energy–momentum tensor, we can find the aforementioned equations for the field's pressure and energy density [10].

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (52)$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad (53)$$

Dividing the two gives us the equation of state for the field [11], ω_ϕ , i.e

$$\omega_\phi = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \quad (54)$$

To model the evolution of the field, we can substitute the above equations for ρ and p in the Friedmann equation while setting $k = 0$, as we are considering a flat universe.

This yields the relation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(V(\phi) + \frac{1}{2}\dot{\phi}^2\right) \quad (55)$$

Through certain mathematical transformations, we can find that the field equation of the inflaton field is defined by

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \quad (56)$$

In physics, systems with a mass that is periodically moving in a predictable manner are called harmonic oscillators. The equation for the system of one highly similar to Equation 57. Here, $\ddot{\phi}$ is the acceleration of the field, $3H\dot{\phi}$ is the velocity of the field and is the Hubble constant, and $V_{,\phi}$ is the derivative of the potential of the field with respect to phi. The Hubble parameter is a value that denotes the expansion of the universe, which is why the velocity of the field is multiplied with it.

For the slow-roll framework, the equations are slightly modified to fit the correct potential curve.

The inflation in the slow-roll model has an extremely flat and long potential. The extensive 'flatness' of the potential curve allows us to essentially eradicate the second derivative of the potential, as

$$\ddot{\phi} \approx 0 \quad (57)$$

This allows us to modify the field equation according to two parameters [12], ε and η given by Equations 58 and 59.

$$\varepsilon = -\frac{\dot{H}}{H^2} \ll 1 \quad (58)$$

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1 \quad (59)$$

where ε tells us that the Hubble parameter must change very slowly with time and η tells us that the curvature of the inflaton potential is an extremely small quantity.

Now that the general dynamics are established, we can explore the modifications made by ultra slow-roll inflation on the existing slow-roll paradigm.

At extremely flat potentials, slow-roll inflation runs into a challenge wherein the kinetic energy falls faster than it would in free-fall. This is not physically possible; which is why ultra slow-roll introduces parameters in which the field acts differently under these conditions.

Since $\ddot{\phi} \approx 0$, we can rearrange the Klein-Gordon equation of the field [13], or Equation 57, to get Equation 60.

$$3H\dot{\phi} \simeq -V_{,\phi} \quad (60)$$

However, as mentioned above, at extremely flat potentials, we must modify the theory slightly. Here, the slope term $V_{,\phi}$ or $\frac{\partial V}{\partial \phi}$ becomes negligible. In this case, the Klein-Gordon equation cand. berearranged to get Equation 61 .

$$\ddot{\phi} + 3H\dot{\phi} \simeq 0 \quad (61)$$

in which case the friction term locks with the acceleration term [14]:

$$\ddot{\phi} \simeq -3H\dot{\phi} \quad (62)$$

Using the two parameters, it is possible to estimate the length and scale of inflation within the slow-roll model in e-folds given by the integral [14]

$$N = \ln\frac{a_{\text{end}}}{a_{\text{beg}}} = \int_{t_{\text{beg}}}^{t_{\text{end}}} H dt \quad (63)$$

To calculate N, we can use an alternative method to arrive to the final value: calculating the size of a Hubble patch during inflation [14]; i.e.

$$H_0^{-1} \times \frac{a_{\text{end}}}{a_0} \times e_{\text{total}}^{-N} \leq H_{\text{inflation}}^{-1} \quad (64)$$

This can be interpreted as taking today's size of a Hubble patch and multiplying it by the growth of the scale factor and a certain number of e-folds. Approximating

$$H^{-1} \approx \frac{1}{T^2} \text{ and a} \approx \frac{1}{T} \quad (65)$$

We can rewrite N as

$$N_{\text{total}} \geq \ln \frac{T_{\text{end}}}{T_0} \approx 60 \qquad (66)$$

We therefore know that for inflation to be valid, we must have at least 60 e-folds [14]. This is a requirement for ultra slow-roll inflation, along with the two parameters ε and η .

This concludes the section's purpose of establishing the necessary parameters for a successful inflationary model of the universe. Taking the same into account, we can now evaluate its weaknesses and suggest possible research avenues to ameliorate the same.

2.2 Results

This section will topically discuss the current weaknesses and challenges in the inflationary model and further suggest pathways of research that can potentially develop the theory to make it more consistent.

2.2.1 Weaknesses of the Inflationary Model

2.2.1.1 The Graceful Exit Problem

The graceful exit problem is the name given to the conceptual challenge of finding a valid model that describes the events following the inflaton potential's transition to the true vacuum.

2.2.1.2 Timeline and Recombination

In the model of slow-roll inflation, the universe is returned to the Big Bang through a process known as recombination, or reheating. As the potential oscillates near the zero-point, potential energy is rapidly converted into kinetic energy and vice versa periodically; and the inflaton potential undergoes a phase transition - all the energy is manifested as relativistic matter at extremely high temperatures [15].

This is caused by a phenomenon known as parametric resonance. The oscillations of the inflaton field potential can resonate constructively with the natural frequencies of other fields, leading to the creation of particles.

Recombination and reheating are also directly connected to the timeline and spectra created by inflation.

This connects directly to Section 2.2.2.1, in which the power spectra of the CMB radiation are key in our verification of warm inflation.

2.2.2 Novel Research Pathways

Having evaluated the weaknesses of the model, it is now possible to investigate the potential pathways of research that can provide insight into solving these challenges.

2.2.2.1 Cosmic Microwave Background

The cosmic microwave background is electromagnetic radiation permeating the universe; or a sea of photons, currently at a temperature of about 2.7 Kelvin [16]. It can be used as a tool for cosmic archaeology. The distribution of the radiation over a range of spectra is essential; as well as the type and nature of the anisotropies in the radiation.

Given the origin of the CMB, it is key in the identification of the necessary initial conditions of the inflationary model as well as models for the guidance of the evolution of the anisotropies during inflation.

The CMB carries imprints from early stages of the universe at extremely high energies. It is a spectrum of density fluctuations that set the initial conditions for the structure of the universe. Exploring the power spectra of primordial gravitational waves that created density fluctuations as well as the resultant power spectrum of the CMB is key in structuring a newer, more accurate framework of inflation.

To map the CMB, we follow the photons pointed at us backwards in their path as they reach the Earth back to the surface where they last scattered from the CMB plasma; i.e. the surface of last scattering. This pattern of photons can be mapped.

As expected, in the map, there should be minute deviations in the density and polarization of the radiation due to the interference of the radiation as it reaches us with the mass structures of the universe [17].

However, in order to give rise to these mass structures, there must have been density perturbations in the early universe. These perturbations lead to anisotropies in the CMB: uneven temperatures and intensities of photons across the map.

To map these perturbations, it is possible to liken the more accessible example of sound waves and air. Sound waves also create density perturbations. If we had a snapshot of the sound in an orchestral room, we can create a map of the density perturbations caused by the sound emitted by each instrument. We can graph the mapped perturbations' frequency against their strength on a graph. The same can be done to the strength of different density waves in the universe [17].

Following the same process outlined above, the most accurate measurement of the CMB is that of the ESA's Planck Satellite.

If we graph the observed photons' brightness against their frequency, the following curve is obtained at a constant temperature of approximately 2.7 Kelvin. This also matches the fact that the anisotropies seen above in the CMB are at the level of only 0.00001 Kelvin, describing a highly uniform CMB [17]. The graph can be seen in Figure 2.

We have to measure the polarization of the light for the next step in analyzing the CMB. This is caused by gravitational waves in the early universe, which are amplified by inflation. They spread in all directions and must be detected along with the photons in the plasma phase. Not only gravitational waves, but also quantum fluctuations create anisotropies in the radiation. Taking into account the effects of both allows us to go further in analyzing the CMB [17]

To set the initial conditions for the fluctuations, the modes can be represented in a Fourier series as they all evolve

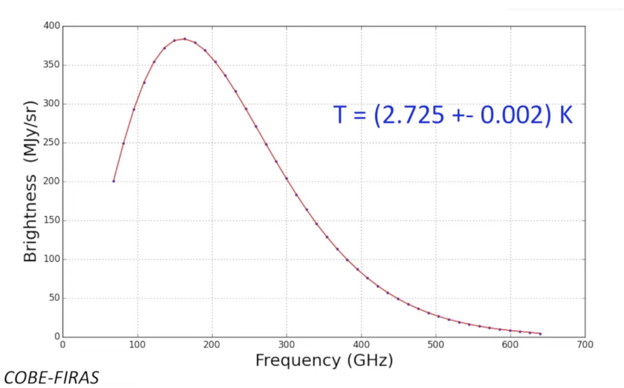

Fig. 2: Cosmic Microwave Background Spectrum[17]

independently. In perturbation theory, the general expression for the change in a value, field, or metric is described as

$$\delta X(t,x) \equiv X(t,x) - \bar{X}(t) \qquad (67)$$

Where $\delta X(t,x)$ are the perturbations, $X(t,x)$ is the field, metric, or value to be perturbed, and $\bar{X}(t)$ is the average background value of the same[18]

The wave modes of the same perturbation fluctuations can be Fourier transformed to become a series of wave modes at a constant wavenumber ; i.e.

$$X_k(t) = \int d^3x X(t,x) e^{ik\cdot x} \qquad (68)$$

The fluctuations in the scalar inflaton field can be treated in the same manner. However, when we take into consideration the fact that the Hubble radius stays constant while the universe expands, it can be postulated that relative to the spacetime manifold, the Hubble sphere shrinks[17].

As the shrinking takes place, the wavelengths of the inflaton's fluctuation modes escape the Hubble radius, thereby stretching out the quantum fluctuations to large scales; and creating the aforementioned density perturbations.

Mathematically, this can be expressed by the inequalities

Superhorizon: $k < aH$ (69)

Subhorizon: $k \gg aH$ (70)

After inflation, as the Hubble sphere begins to grow, the radius will catch up with k. The fluctuations result in two classes of perturbations; i.e. scalar and tensor. Scalar perturbations are changes to the curvature, defined by

$$\Re = \psi + \frac{H}{\dot{\phi}}\delta\phi \qquad (71)$$

while tensor perturbations caused by gravitational waves are defined by

$$h^i_i = \partial^i h_{ij} = 0 \qquad (72)$$

which is a transverse, traceless perturbation in the spacetime metric formalized by a tensor[19].

With the formalization of these two perturbations, we can logically derive the fact that they are equivalent at horizon exit and at causal horizon re-entry[18]. The figure below graphs the constant density fluctuations of the universe against the size of the comoving horizons, once again on comoving scales, against time on a logarithmic scale, as seen in Figure 3.

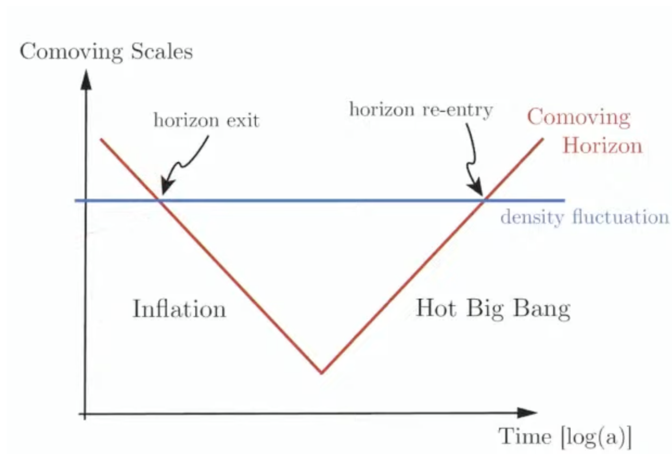

Fig. 3: Comoving Horizon Scale Graph[18]

Since we know that the density fluctuations are the same at exit and re-entry, we can use the initial conditions of the CMB to make predictions about post-inflationary physics.

The vertex of the comoving causal horizon graph is analogous to the reheating stage of inflation[18], as shown in Figure 4.

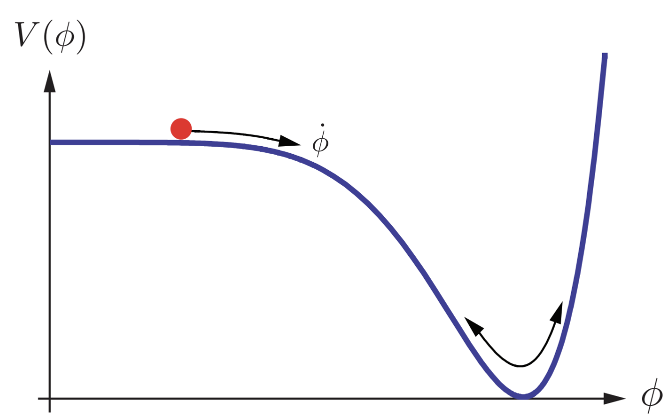

Fig. 4: Inflaton Field Potential[20]

As the field oscillates at its vacuum point, a process known as parametric resonance occurs in which, due to its coupling with other fields, a gargantuan amount of radiation and matter is produced by the inflaton as it resonates with other fields. This produces a universe filled with matter and radiation at energy levels of the Big Bang[18].

At this stage, the density fluctuations are identical to what they were at the exit of the wavemodes from the Hubble horizon.

Because of the constant nature of the density fluctuations, we can extrapolate the conditions at horizon re-entry to today's universe.

These theoretical predictions at both phases of inflation can then be cross-referenced with observations and thereby filter out theories that do not pass the heuristic test.

2.2.2.2 Galactic Redshift Analysis

In this section, we will explore a novel potential pathway of research for the verification or negation of a wide variety of inflationary models.

For this, we will be using more conventional evolutionary cosmology; particularly galactic and stellar formation as well as redshift and the expansion of the universe.

The James Webb telescope, a pioneer of the same field, released a set of astonishing images in 2022 of the early universe.

By spectrometry and photoanalysis, it is possible to calculate the redshift of the galaxies based on their distance from us as observers[21], allowing for the derivation of the Hubble constant, or Hubble rate. The inverse of the Hubble rate will then give us the age of the universe; which can be used to corroborate various models of inflation based on their time period, or length of inflation in e-folds.

We will first derive a generalized formula or construct for the age of the universe based on estimations from the redshift of the JWST-imaged galaxies; then substituting it into the relevant estimations of age through inflationary dynamics.

3 Discussion

The literature review undertaken above yields great insight into the status quo of inflationary cosmology in the 21st Century. Through the analysis of old inflation, slow-roll inflation, and ultra slow-roll inflation, a number of weaknesses – both conceptual and mathematical – were evaluated. Through further review, two primary research methods were discussed in order to address these weaknesses.

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