

# Combinatorics and Group Theory in Chess and Music

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Human beings have engaged in the practice of counting since ancient times, as evidenced by various historical records. While early methods likely involved simple techniques such as counting on fingers, over time, these methods have evolved into sophisticated mathematical concepts. This paper explores a version of counting problems, focusing on how combinatorics and group theory can be applied to interesting challenges in chess and music—two domains that have been integral to human culture since the earliest civilizations and subjects of mathematical inquiry for centuries. The paper is divided into two sections. The first section uses elementary combinatorics to determine the maximum number of identical chess pieces that can be positioned on a chessboard without conflicting with one another. We then calculate the minimum number of pieces required to dominate all the squares on the board. The second section applies group theory to solve two problems: first, determining the number of ways to color the faces of a cube using three colors, accounting for different rotational symmetries; and second, calculating the number of unique four-note chords using Burnside's lemma. The motivation behind selecting these problems is not only to explore the mathematical structures underlying chess and music but also to demonstrate how counting problems can be applied to elements of everyday life. These concepts can be extended to fields such as theoretical computer science, other strategy games like Go, or different applications of Burnside's lemma in music theory.

## Introduction

Humans have explored counting since the very early times<sup>1</sup>. We probably first started counting by using fingers, and then by mapping using beads, symbols, markings and then became more sophisticated<sup>2</sup>. We, humans, applied the counting problem to our everyday circumstances like taking care of food supplies, sending messages, trading between villages and even keeping track of how many animals were there in livestock<sup>3</sup>. In this paper we explore a version of counting problems - how combinatorics and group theory can be used to solve interesting problems in chess and music.

To manage counting, various approaches are taken through enumerative combinatorics<sup>4,5,6</sup>, graph theory<sup>7,8</sup>, statistics<sup>9</sup> and probability<sup>10</sup>, among others. Combinatorics have been used by the earliest civilizations to effectively and succinctly count<sup>11</sup>. Similarly, group theory<sup>12</sup> and Burnside's lemma<sup>13</sup> have found its applications in numerous fields like Physics<sup>14</sup>, Chemistry<sup>15</sup>, Quantum mechanics<sup>16</sup> and so on. Burnside's lemma addresses counting problems in symmetric systems<sup>17</sup>. Music and chess have been an integral part of human culture from very early civilizations<sup>18,19</sup>. These fields have also been a focus of mathematical enquiry for long. The relationship between maths and chess have been explored in<sup>20</sup> and<sup>21</sup>. Similarly, application of math in music has also been looked at by many scholars<sup>22,23</sup>.

In this paper we will see the application of combinatorics and group theory to some theoretical but interesting counting problems around chess and music. In the first section, we apply

combinatorics to answer questions like how many identical chess pieces can be positioned on a chessboard without conflicting with one another. We also look at the minimum number of the same piece needed to dominate all the squares on the board. While these problems might not have any practical application, they allow to understand the mathematical structure behind chess. In the second section, we explore group theory to arrive at a more advanced combinatorial tool - the Burnside's lemma. We naturally arrive at the lemma by trying to calculate the number of ways to colour the faces of a cube using three colours, where two colourings identified by a rotation are considered equivalent. Following that, we apply Burnside's lemma to music and calculate the number of unique chords with four notes. These again allow us to understand the mathematical structure behind music.

## Combinatorics used in chess

In this first section, we apply elementary combinatorics to determine the maximum number of identical chess pieces that can be positioned on a chessboard without conflicting with one another. We then calculate the minimum number of the same piece needed to dominate all the squares on the board.

### Maximum number of pieces

In this section, we find the answer to the question of what is the maximum number of chess pieces of the same kind that can be

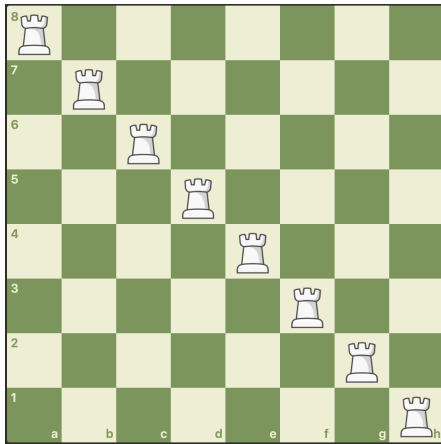


Figure 2.1: Maximum number of rooks

placed on a standard chess board so that they cannot target each other. We start with the easier cases such as rooks and knights, then move onto bishops and kings and finally cover the case of queens.

**Proposition 2.1.** The maximum number of rooks that one can put on the chessboard so that they cannot take each other is equal to eight.

*Proof.* An example with eight rooks is given in Figure 2.1. If there are more than eight rooks, there are at least two rooks in the same row, which means that they can take each other.  $\square$

**Proposition 2.2.** The number of possible arrangements of 8 identical rooks on the chessboard so that they cannot take each other is equal to  $8!$ .

*Proof.* On an empty chessboard, to place the first rook on the first row, we have eight options. Since we cannot have more than two rooks in the same column or row, on the second row, we have seven options to place the rook. On the third row, we have six options, and so on. So the number of possible arrangements of eight identical rooks is equal to

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8! \quad \square$$

**Proposition 2.3.** The maximum number of knights that one can put on the chessboard so that they cannot take each other is equal to 32.

*Proof.* A knight placed on a white square can only move to black squares. So if all the knights are placed on the white squares, then the maximum number of knights is  $\frac{64}{2} = 32$ . An example with 32 knights is given in Figure 2.2.

Let us show that, in general, it is not possible to place more than 32 knights on the chessboard so that they cannot take each other. First, consider Figure 2.3 below, which has eight squares. We can only fit four knights in these eight squares so that they cannot take each other. Subdividing the chessboard into the  $4 \times 2$  parts as in Figure 2.3, there is a maximum of 32 knights that we can place so that they cannot take each other.

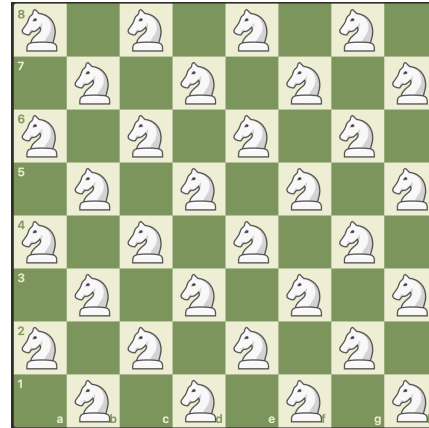


Figure 2.2: Maximum number of knights



Figure 2.3: Maximum number of knights

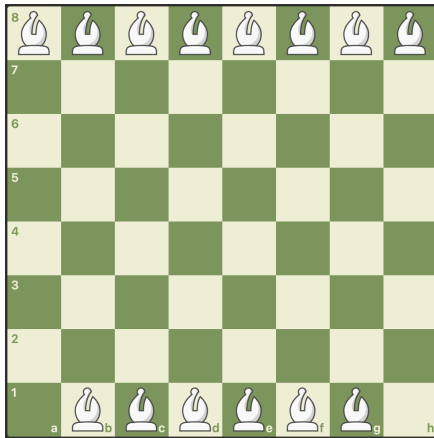


Figure 2.4: Maximum number of bishops

**Proposition 2.4.** The number of possible arrangements of 32 knights on the chessboard so that they cannot take each other is equal to two.

**Proof.** Consider a knight's tour of the chessboard. For details refer to<sup>24</sup>. A knight's tour is a sequence of moves by a knight on a chessboard such that the knight visits every square exactly once. Obviously, if 32 of the 64 squares are occupied by non-attacking knights, then the tour must alternate between occupied and unoccupied squares, since a horse can only attack a square of the opposite colour. This means that there are (at most) two ways to place the 32 knights. We can offer examples in these two ways: namely by placing all the knights on the white squares or all on the black squares. □

**Proposition 2.5.** The maximum number of bishops that one can put on the chessboard so that they cannot take each other is equal to 14.

**Proof.** An example with 14 bishops is given in Figure 2.4. Referring to Figures 2.5 and 2.6, we can see that there are seven white diagonals labelled w1-w7 and eight green diagonals labelled b1-b8. Due to b1 and b8 being at the opposite ends, a bishop placed at either of those squares covers the other, leaving the remaining fourteen diagonals uncovered. We start placing the bishops to cover the green diagonals, and we can see that there is only one way to cover b1. Now coming to b2, which has three squares, we see that there are two squares where we can place the bishop, and also two squares to place the bishop in b7 as well. Upon placing the bishop in either square on b2, it leaves only one unoccupied square on the diagonal b7. Now coming to b3 and b6, out of the five squares, there are only two squares where we can place the bishop so that the bishops cannot take each other. This way, we can fit only 14 bishops that cannot capture one another. □

**Proposition 2.6.** The number of possible arrangements of 14 bishops on the chessboard so that they cannot take each other is 128.

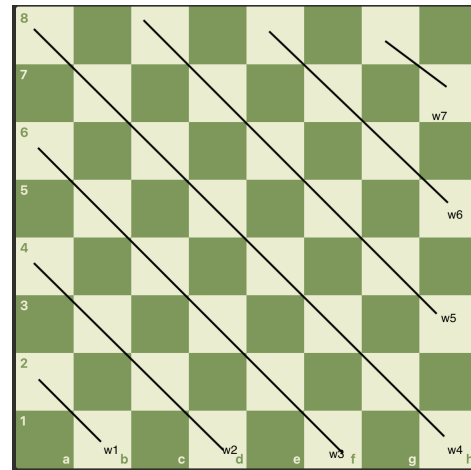


Figure 2.5: White diagonals

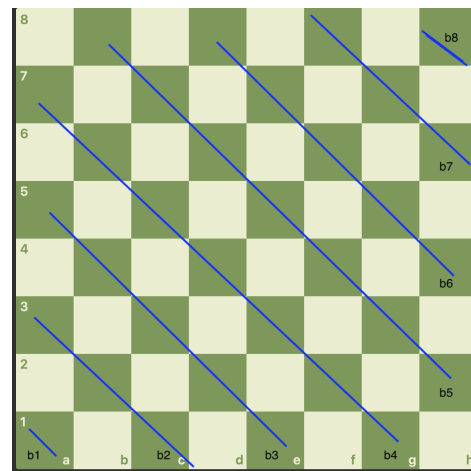


Figure 2.6: Green diagonals

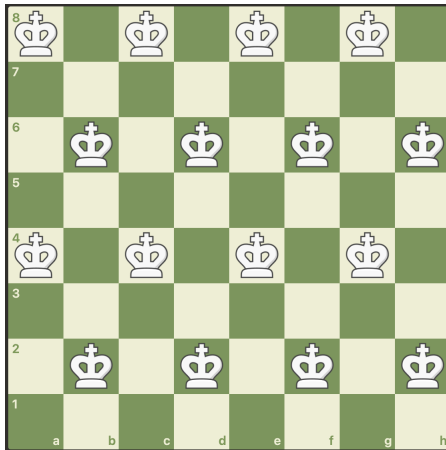


Figure 2.7: Maximum number of kings

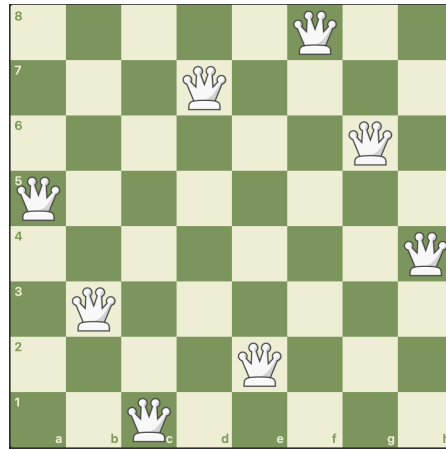


Figure 2.8: Maximum number of queens

Proof. Referring to Figures 2.5 and 2.6, we see that for the corner diagonal, which is b1, we have only one option to place the bishop, and as we move on (to w1, b2, etc.), the options reduce to two for each diagonal. Hence, we have  $1 \cdot 2 \cdot 2 \cdot 2 = 8$  ways to place bishops on the green diagonals, and similarly, we have  $2 \cdot 2 \cdot 2 \cdot 2 = 16$  ways to place bishops on the white diagonals. So the different number of ways to place the bishops is  $8 \cdot 16 = 128$ . We refer to<sup>25</sup> for details.  $\square$

**Proposition 2.7.** The maximum number of kings that one can put on the chessboard so that they cannot take each other is equal to 16.

Proof. An example with 16 kings is given in Figure 2.7. A king can move one square in any direction (vertical, horizontal, and diagonal). Upon subdividing the board into two by two squares, each of the subdivisions has one king at most. Hence, we have 16 subdivisions, with one king at the most. Any more than one king in the subdivisions results in the king being targeted. Hence, only 16 kings can fit, and more than 16 kings can take each other.  $\square$

As far as we are aware, there is no known solution to the question of the number of possible arrangements of 16 kings on the chessboard so that they cannot take each other. Rather, this has been calculated using an enumeration of all possible steps using a computer algorithm. It is clear that one can calculate it on the computer.

**Proposition 2.8.** The maximum number of queens that one can put on the chessboard so that they cannot take each other is equal to eight.

Proof. An example with eight queens is given in Figure 2.8. A queen can move in any direction (vertical, horizontal, and diagonal). It is clear we cannot have more than eight queens as we cannot have more than eight rooks, and the queen has all the capabilities of the rook. Hence, only eight queens can fit and more than eight queens can take each other.  $\square$

**Proposition 2.9.** The number of possible arrangements of

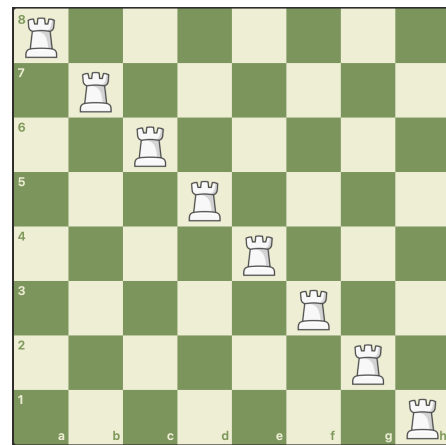


Figure 2.9: Minimum number of rooks

eight queens on the chessboard, also known as the eight queens puzzle, has 92 distinct solutions. If solutions that differ only by the symmetry operations of rotation and reflection of the board are counted as one, the puzzle has 12 solutions.

Proof. For the proof we refer to<sup>26</sup>

### Minimum number of pieces

In this section, we find the answer to the question of how many minimum numbers of pieces can be placed on a standard chess board so that they control all the squares.

**Proposition 2.10.** The minimum number of rooks that one can put on the chessboard so that they can cover all the squares on the chessboard is eight.

Proof. An example with eight rooks is given in Figure 2.9. This is because there are eight rows and columns, and we need one rook to control a row and a column. With seven rooks, there are one row and one column that are not controlled. Hence, the minimum number of rooks needed is eight.  $\square$

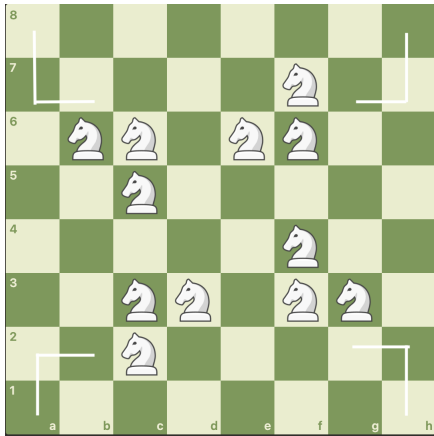


Figure 2.10: Minimum number of knights

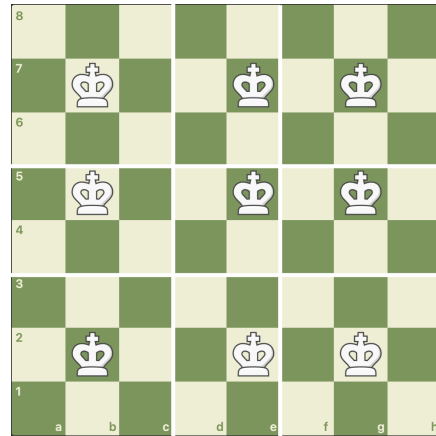


Figure 2.12: Minimum number of kings

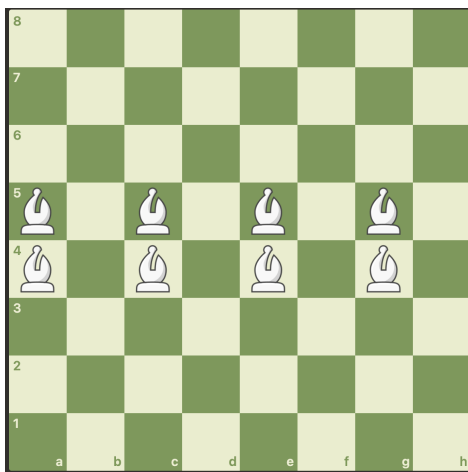


Figure 2.11: Minimum number of bishops

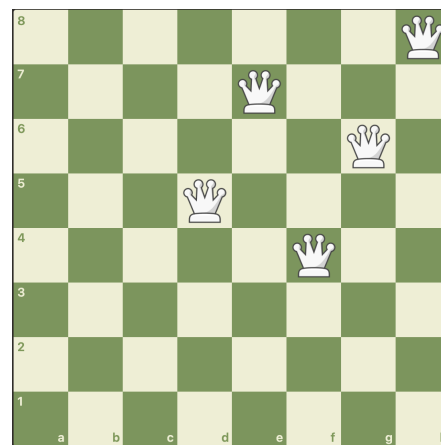


Figure 2.13: Minimum number of Queens

**Proposition 2.11.** The minimum number of knights that one can put on the chessboard so that they cover all the squares on the chessboard is 12.

Proof. Indeed, considering the 12 squares marked by a white line, a single knight cannot cover more than one of those squares. Thus, we need at least 12 knights to cover those 12 squares. With 12 knights, we can actually attack the whole board, as seen in Figure 2.10. Hence, the minimum number of knights needed is 12. We refer to<sup>27</sup> for details. □

**Proposition 2.12.** The minimum number of bishops that one can put on the chessboard so that they cover all the squares on the chessboard is eight.

Proof. An example with eight bishops is given in Figure 2.11. This is because there are 15 diagonals, but the corners of the chessboard are part of the same diagonal. So each of the bishops controls one diagonal, while the two bishops at the centre control two each, so four diagonals in total. Hence, the minimum number of bishops needed is eight. □

**Proposition 2.13.** The minimum number of kings that one

can put on the chessboard so that they cover all the squares on the chessboard is nine.

Proof. An example with nine kings is given in Figure 2.12. As seen in the picture, the board is divided into nine sections. With the corners of the board having nine squares in one section while the middle of the sides having six squares. As a king can cover a maximum of eight squares, excluding the one it is on, there is a minimum of nine kings required. □

**Proposition 2.14.** The minimum number of queens that one can put on the chessboard so that they can cover all the squares on the chessboard is five.

Proof. An example with five queens is given in Figure 2.13. Each of the five queens controls one row and one column, leaving only nine squares, which are covered by the queen's diagonal movement. The knight position placement of the queens helps in covering different diagonals. To show that four queens are not possible, we enumerate using a computer and to do that one needs a computer algorithm. We refer to<sup>28</sup> for details □

## Group Theory

In this section, we use elementary group theory<sup>29</sup> to first answer the question:

In a cube, if we have three colour options to colour the faces, the number of different possibilities we have is 36. So how many of the 36 colourings do not change when we apply rotations of the cube?

To answer this question, we need to study the rotations of a cube systematically. Doing so, we arrive at group theory and discuss Burnside's lemma. It provides a formula to count the number of objects, where two objects that are symmetric by rotation or reflection are not considered the same. After this, we discuss the application of the Burnside lemma in music theory and answer a question: How many distinct tetrachords are possible?

### Number of rotations

Counting the possible orientations of the cube, we know that there are 24 rotations. To summarize, the rotations are:

1. One identity rotation that does nothing.
2. Nine rotations around axes through the middle of one of the three pairs of opposing faces, with rotations of  $90^\circ$ ,  $180^\circ$  and  $270^\circ$ .
3. Six rotations around axes through the middle of one of the six pairs of opposing edges, with a rotation of  $180^\circ$ .
4. Eight rotations around axes through one of four pairs of opposing corners, with rotations of  $120^\circ$  and  $240^\circ$ .

Let us look at these in more detail:

1. First is the identity rotation, when we have no rotation.
2. Second is the nine rotations of the cube about the three axes shown in Figure 3.1 below. We can either rotate by  $90^\circ$ ,  $180^\circ$  or  $270^\circ$  around either the red, blue or green axes. Each of these rotations leave two faces fixed, and all vertices and edges are not fixed.

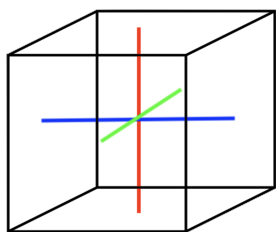


Figure 3.1: Rotation 2

3. Third, we have the six rotations of the cube by  $180^\circ$  around the six axes shown in the Figure 3.2 below. Each of these rotations leave two edges fixed and no faces or vertices fixed.

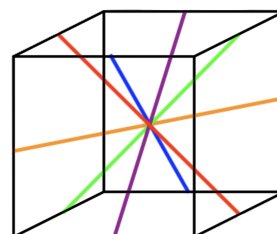


Figure 3.2: Rotation 3

4. Finally, we have rotations by  $120^\circ$  and  $240^\circ$  around the four axes shown in Figure 3.3 below. This gives eight rotations, each of which leaves two vertices fixed and no edges or faces fixed.

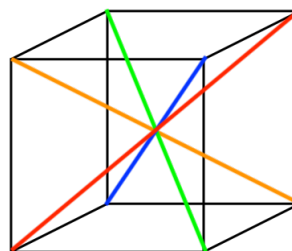


Figure 3.3: Rotation 4

Hence, there are a total of  $1 + 9 + 6 + 8 = 24$  rotations.

We make a few observations about the rotations of the cube. First, composing two rotations gives a rotation. Second, composing three rotations can be done in any order. Third, there is also a rotation such that composing it with any other rotation is equal to that rotation. Fourth, for any rotation, there is also an inverse rotation. We look at this in detail later in the paper. This leads to the following abstract definition:

**Definition 3.1.** A group is a set with an operation such that if any two of its elements are combined through the operation it produces a third element belonging to the same set (the set is closed under the operation) and meets the following three properties: associativity, existence of identity and existence of inverse. Suppose  $\text{Dot}(\cdot)$  is an operation and  $G$  is the group, then the axioms of group theory are defined as:

- **Associativity:** For all  $a, b, c \in G$ , we have  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ .
- **Existence of Identity:** There exists  $e \in G$  such that  $e \cdot a = a = a \cdot e$  for all  $a \in G$ .

- **Existence of Inverse:** For all  $a \in G$ , there exists  $a^{-1} \in G$  such that  $a \cdot a^{-1} = e = a^{-1} \cdot a$ .

If the operation also satisfies

- **Commutativity:** For all  $a, b \in G$ , we have  $a \cdot b = b \cdot a$ , then  $G$  is called a commutative group.

A few examples of groups are integers (commutative), remainders (commutative), and rotations of a cube (not commutative).

• **Integers**

1. **Associativity:** For integers, addition is associative. For any integers  $a, b$ , and  $c$ ,  $(a + b) + c = a + (b + c)$ .
2. **Existence of Identity:** The identity element is 0 since for any integer  $a$ ,  $a + 0 = 0 + a = a$ .
3. **Existence of Inverse:** For any integer  $a$ , its inverse (additive inverse) is  $-a$  since  $a + (-a) = (-a) + a = 0$ .
4. **Commutativity:** For any integers  $a$  and  $b$ ,  $a + b = b + a$ .

- **Remainders** Let us consider remainders modulo  $n$  as a group under addition (mod  $n$ ).

1. **Associativity:** In modular arithmetic, addition is also associative for remainders. For any remainders  $a, b$ , and  $c$ ,  $(a + b) + c \equiv a + (b + c) \pmod{n}$ .
2. **Existence of Identity:** The identity element is 0 since for any remainder  $a$ ,  $a + 0 \equiv 0 + a \equiv a \pmod{n}$ .
3. **Existence of Inverse:** For any remainder  $a$ , its inverse exists. For example  $n - a$  is its inverse.
4. **Commutativity:** For remainders  $a$  and  $b$  modulo  $n$ ,  $a + b \equiv b + a \pmod{n}$ .

• **Rotations of a Cube**

1. **Associativity:** Rotations of a cube follow the associative property. For instance, rotating first by  $90^\circ$  along one axis and then by  $180^\circ$  along another axis yields the same result regardless of the order of operations.
2. **Existence of Identity:** The identity element is the “no rotation” operation, where the cube remains unchanged after the rotation.
3. **Existence of Inverse:** Each rotation has an inverse rotation that undoes its effect. For example, a  $90^\circ$  rotation can be undone by another  $90^\circ$  rotation in the opposite direction.

4. **Non-commutativity of Cube Rotations:** From Figure 3.4, we can see that applying the rotation across the  $z$ -axis and then the  $x$ -axis is not the same as rotating along the  $x$ -axis first and then the  $z$ -axis. This shows us that the rotations of a cube that do not follow commutativity are not commutative.

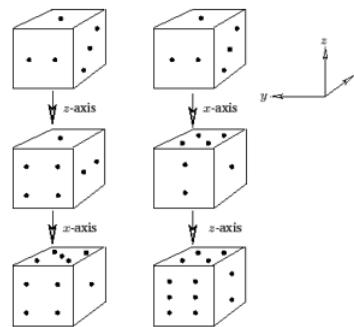


Figure 3.4: Non-commutativity of a cube

**Group Action**

We can make a few observations about how the rotations of a cube affect the colouring. Suppose there is a group  $G$ , which is a group of all the rotations of the cube, and we have a set  $X$  which is the set of all the possible 36 colourings of the cube. If we apply a rotation  $g(\in G)$  to a colouring  $x(\in X)$ , then we get another colouring  $g \cdot x$ , a new colouring and element of the set  $X$ . Then we observe that this has a few properties:

- If we rotate a colouring by  $g$  and then by another rotation  $h$ , it is the same as rotating the colouring by the composition of the two rotations  $h \circ g : h \cdot (g \cdot x) = (h \circ g) \cdot x$
- If we take the identity rotation  $e$  and apply this to any colouring  $x$ , it does not change the colouring:

$$e \cdot x = x$$

**Definition 3.2.** A group action of a group  $G$  on a set  $X$  is a map  $G \times X \rightarrow X$  which satisfies the two properties above<sup>29</sup>.

**Burnside’s Lemma**

Burnside’s Lemma is a result of group theory that can be used to count distinct objects with respect to symmetry. In particular, it provides a formula to count the number of objects, where two objects that are symmetric by rotation or reflection are categorised as the same.

To count the colourings of the cube, we use Burnside's Lemma. However, we need to introduce some more terminology before that.

**Definition 3.3.** For a group  $G$  acting on a set  $X$  and for any element  $x \in X$  we say that the orbit of  $x$   $\text{Orb}(x)$  is the set of all elements of  $X$  of the form  $g \cdot x$  for  $g \in G$ . One can check that for any two elements  $x, y \in X$  either  $\text{Orb}(x) = \text{Orb}(y)$  or the orbits do not intersect. Therefore, the whole  $X$  is partitioned into a set of orbits, which we denote by  $\text{Orb}(X)$ .

**Definition 3.4.** A fixed point of an element  $g \in G$  is an element  $x \in X$  such that  $g \cdot x = x$ . The set of all elements  $x \in X$  that are fixed by  $g \in G$  is denoted by  $\text{Fix}(g)$ . We are ready to state Burnside's lemma.

**Definition 3.5.** Let us say we have a group action of group  $G$  on a set  $X$  (in particular,  $G$  is the group of rotations and  $X$  the set of the 36 colourings). Then there is the following identity:

$$|\text{Orb}(X)| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

where we use the notation  $|Y|$  to denote the number of elements on a set  $Y$ . This means the number of orbits is equal to the average number of fixed points in  $G$ .

Proof. We refer to<sup>29</sup> for the proof of the Burnside's lemma.  $\square$

**Proposition 3.6.** In a cube, if we have three colour options to colour the faces, the number of different possibilities we have is  $3^6$ . However, the number of different colourings of the faces of the cubes onto three colours where the two colourings identified by a rotation are considered the same is equal to 57.

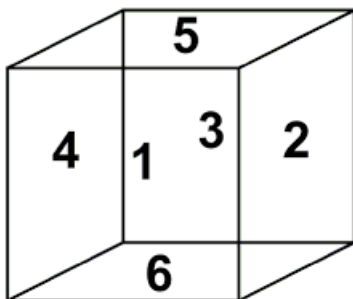


Figure 3.5: Cube

Proof We prove the result by applying Burnside's lemma. Therefore, we need to calculate  $|\text{Fix}(g)|$  for every rotation  $g$  of the cube. We use Figure 3.5 to explain the computation. We know from an earlier discussion that there are 24 different rotations. Let us consider each type of rotation.

- In rotation  $e$ , that is the identity rotation or do nothing rotation. Each face has three options for colouring:

$$\text{Fix}(e) = 3^6$$

- Rotation by  $90^\circ$  along the line connecting opposite faces (4 and 2) - Figure 3.1. With reference to Figure 3.5, the faces 4 and 2 remain fixed. While 5, 1, 6, 3 upon rotation become 3, 5, 1, 6. Hence, those four sides are of the same colour. Since there are six ways we can perform this rotation (along the three axes in anticlockwise and clockwise directions), we have:

$$\text{Fix}(90^\circ) = 6 \cdot 3^3$$

- In rotation by  $180^\circ$  along the line connecting opposite faces (4 and 2) - Figure 3.1. With reference to Figure 3.5, the faces 4 and 2 remain fixed. While 5, 1, 6, 3 upon rotation become 6, 3, 5, 1. Hence, faces 6 and 5 swap places, and faces 1 and 3 swap. Since there are three ways we can perform this rotation (along the three axes, the direction of the rotation does not matter). Hence,

$$\text{Fix}(180^\circ) = 3 \cdot 3^4$$

- In rotation by  $180^\circ$  along the line connecting the midpoints of opposite edges (midpoint of the edge of the faces 4 and 1; 2 and 3) - Figure 3.2. With reference to Figure 3.5, each pair of opposite edges swap. Faces 1 and 4; 2 and 3; and 5 and 6 swap. There are six ways we can perform this rotation (along the six axes, direction does not matter). Hence,

$$\text{Fix}(180^\circ) = 6 \cdot 3^3$$

- Rotation by  $120^\circ$  along the line connecting opposite vertices (the vertex connecting faces 4, 1, 6, and vertex connecting faces 2, 5, 3) - Figure 3.3. With reference to Figure 3.5, faces 4, 6, and 1 swap places, and faces 2, 5, and 3 swap. Since there are eight ways we can perform this rotation (along the four axes, in clockwise and anticlockwise directions). Hence,

$$\text{Fix}(120^\circ) = 8 \cdot 3^2$$

Applying Burnside's lemma, we get

$$|\text{Orb}(X)| = \frac{1}{24} (3^6 + 6(3^3) + 3(3^4) + 6(3^3) + 8(3^2)) = \frac{1368}{24} = 57$$

$\square$

### Using Burnside's lemma in Music

We can also apply this theorem to music theory in the enumeration of distinct chords. This is useful to identify patterns within chord sequences, in addition to counting the number of patterns in intervals and rhythms. It can also be helpful in transposing music if transpositions can be easily identified. In Western music, tones are equivalent if they differ by an octave.

There are twelve tones: C, D, E, F, G, A, and B, plus five flats and equivalent sharps in between. C sharp/D flat, D sharp/E flat, F sharp/G flat, G sharp/A flat, and A sharp/B flat. In Figure 3.6 the tones have been numbered from 0 to 11

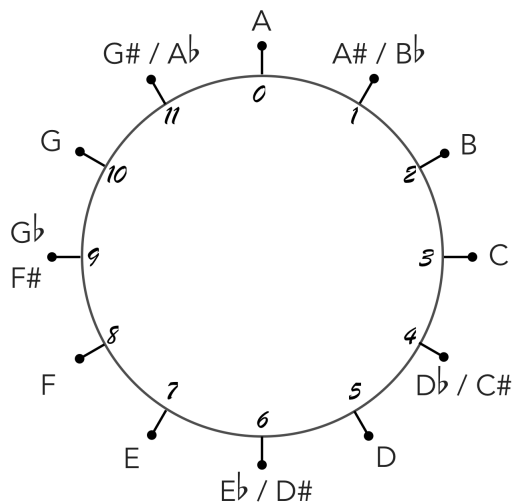


Figure 3.6: Circle of 12 tones, numbered 0 to 11

Remark 3.7. A set of four notes is known as a tetrachord. We consider two tetrachords equivalent if the tones are permuted within the same set of notes or if one can be translated to another.

If the tones are permuted with the same set of notes, for example,  $[C, E, F, G, ] = [3, 7, 8, 10]$  is the same as the chord  $[G, F, C, E] = [10, 8, 3, 7]$  because they consist of the same notes, just in a different order. This is called an inversion.

If one can be translated into another, for example, the C major chords  $[C, E, EG] = [3, 7, 8, 10]$  and  $[E^b, G, A^b, B^b] = [6, 10, 11, 1]$  are considered equivalent because it is a shift of three tones above each note in the first chord. Thus, if the spaces between the tetrachords are equal, then the chord has simply been translated.

In this section, we are going to use Burnside's lemma to answer the question: How many different groups of tetrachords can we make such that the tetra-chords within the same group are all equivalent?

Let us denote the set of all distinct tetrachords of elements as  $X$ , where  $X$  is composed of selections of four unique tones from a set of 12 tones.

First, we account for the equivalence due to permutations. The number of ways to select four tones from a set of 12 without considering ordering is  $|X| = \binom{12}{4}$ , which eliminates the problem of different chord permutations and does not count these multiple times.

Also, let us assume that the tetra-chord is  $(a, b, c, d)$ . Given that a re-ordering results in equivalence, we can safely assume that  $a < b < c < d$ .

Consider the group action where any number  $k$  acts (translates) on the tones  $(a, b, c, d)$  of a tetrachord:

$$k \cdot (a, b, c, d) = (a+k, b+k, c+k, d+k)$$

Here,  $k$  can take any value within the set of remainders (mod 12).  $G$  is defined as the translation of chords by tone  $k$ . Now we use Burnside's lemma to calculate the number of distinct groups of tetrachords. As we saw earlier:

- (i) There is some translation which allows a tetrachord in one group to be translated to any other tetrachord within the same group.
- (ii) There is no translation which can translate a tetrachord in a group to a tetrachord in another group.

Let us apply the Burnside's Lemma, taking into account the  $Fix(k), k \in Z_{12}$  Assume  $(a, b, c, d)$  is a tetrachord where  $a < b < c < d$ , which when translated by  $k$  results in the same chord.

For a non-zero  $k$ , each of the elements needs to change its place. So we need to consider all permutations such that no element is in its original position. So there are 9 possibilities:

$$\begin{aligned} &(b, a, d, c), (d, a, b, c), (c, a, d, b) \\ &(b, d, a, c), (d, c, a, b), (c, d, a, b) \\ &(c, d, b, a), (b, c, d, a), (c, b, d, a) \end{aligned}$$

Let us first consider the three possibilities due to cyclic displacement, which are (huvdad), (codah), and (deble), Let us call them Case I, Case II, and Case III, respectively. Later, we show that all the nine possibilities can be mapped to these three.

Case I: When, suppose we get  $(b, c, d, a)$  after translation by  $k$ . Then we have:

$$a+k=b, \quad b+k=c, \quad c+k=d, \quad d+k=a$$

This gives us  $4k=0$  and  $(b=a+k, c=a+2k, d=a+3k)$ . Which implies that  $k=3, 6, 9$

Case II: When, suppose we get  $(c, d, a, b)$  after translation by  $k$ . Then we have:

$$a+k=c, \quad b+k=d, \quad c+k=a, \quad d+k=b$$

This gives us  $2k=0, a$  and  $b$  are independent and  $(c=a+k, d=b+k)$  Which implies that  $k=6$

Case III: When, suppose we get  $(d, a, b, c)$  after translation by  $k$ .

$$a+k=d, \quad b+k=a, \quad c+k=b, \quad d+k=c$$

This is nothing but a variation of Case I, where all the four elements have some relationship with each other (a difference of a multiple of three) and, therefore, does not result in any unique tetrachords.

Similarly, we can show that all the nine variations can be mapped to one of the above two cases. Either all the variables are related (like Case I) or there are two independent sets (like Case II). For all the other values of  $k(k = 1, 2, 4, 5, 7, 8, 10, 11)$ ,  $\text{Fix}(k) = 0$ . Let us now consider each  $k = 0, 3, 6, 9$ .

$$k = 0$$

The numbers remain the same after permutation. So we have  $\binom{12}{4}$  possibilities.

$$\text{Fix}(0) = \binom{12}{4}$$

Case I:  $k = 3, (b = a + k, c = a + 2k, d = a + 3k)$

If  $a = 0$ , then  $a, b, c, d = 0, 3, 6, 9$ .

If  $a = 1$ , then  $a, b, c, d = 1, 4, 7, 10$ .

If  $a = 2$ , then  $a, b, c, d = 2, 5, 8, 11$ .

When the value of  $a$  is equal to or greater than three, the values start to repeat. The order of the values may change, but the actual values themselves are the same.

$$\text{Fix}(3) = 3$$

$$k = 9$$

The treatment for  $k = 9$  is similar to that of  $k = 3$

$$\text{Fix}(9) = 3$$

$$k = 6$$

Case I:  $(b = a + k, c = a + 2k, d = a + 3k)$  Here,  $c = a + 2k = a + 12 = a$ . Therefore, this is a contradiction.

Case II: (i)  $a$  and  $b$  are independent, (ii)  $(c = a + k, d = b + k)$

When  $a = 0, b = 1 \dots 5$

$(0, 1, 6, 7)(0, 2, 6, 8) \dots (0, 5, 6, 11)$

When  $a = 1, b = 2 \dots 5$

$(1, 2, 7, 8)(1, 3, 7, 9) \dots (1, 5, 7, 11)$

When  $a = 2, b = 3, 4, 5$

$(2, 3, 8, 9)(2, 4, 8, 10)(2, 5, 8, 11)$

When  $a = 3, b = 4, 5$

$(3, 4, 9, 10)(3, 5, 9, 11)$

When  $a = 4, b = 5(4, 5, 10, 11)$

$\text{Fix}(6) = 5 + 4 + 3 + 2 + 1 = 15$

Now upon using Burnside's Lemma we get

$$|\text{Orb}(X)| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|,$$

$$|\text{Orb}(X)| = \frac{1}{|G|} \left( \binom{12}{4} + 3 + 3 + 15 \right)$$

$$|\text{Orb}(X)| = \frac{1}{12} \left( \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} + 3 + 3 + 15 \right)$$

$$|\text{Orb}(X)| = \frac{1}{12} (495 + 21)$$

$$|\text{Orb}(X)| = \frac{1}{12} \times 516$$

$$|\text{Orb}(X)| = 43$$

This represents the number of distinct tetrachords.

## Conclusion

In this paper we explored the counting problem and some of the mathematical aspects of chess and music. In the first section, we applied combinatorics to determine the maximum number of identical chess pieces that can be positioned on a chessboard without conflicting with one another. We then solved questions around the minimum number of the same piece needed to dominate all the squares on the board using the principles of combinatorics. In the second section, we explored group theory to arrive at a more advanced combinatorial tool - the Burnside's lemma. We arrived at the lemma by trying to calculate the number of ways to colour the faces of a cube using three colours, where two colourings identified by a rotation are considered equivalent. Additionally, we explored tetrachords in music and calculated the number of unique tetrachords by applying Burnside's lemma.

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