Diophantine Approximation Using Farey Sequences

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Farey fractions have long been a subject of interest to mathematicians since their classification in 1816, providing a convenient way to order fractions and often showing up in unexpected places in Number Theory. Diophantine Approximation, the study of approximating irrational numbers with rational numbers, has many practical uses, including in public key cryptography (where it provides a method to attack the well-known RSA Cryptosystem) and the study of transcendental numbers. In this paper, we investigate the ties between Farey Sequences and Diophantine Approximation. We cover elementary results relating to Farey Fractions that become central to later parts of the paper. We then link the notion of Farey Sequences with the study of Diophantine Approximation. We build up to Hurwitz's Approximation Theorem, known to be an optimal theorem for approximating irrational numbers with rationals, using only Farey Fractions, and then discuss further improvements on Hurwitz's Theorem with a few adjustments. We continue this discussion via the natural extension to the Lagrange Spectrum, where we discuss the significance of Lagrange Numbers, their connection with Markov Numbers, and a few elementary results. Finally, we present a geometric interpretation of Farey Fractions known as Ford Circles, which can also be used to illustrate weaker Diophantine Approximation results.

Introduction

The name Farey fractions[1] comes from the English geologist John Farey Sr., who wrote a letter about these sequences that was published in the Philosophical Magazine in 1816. Interestingly enough, another mathematician, Charles Haros, independently published similar results in 1802, which was not known to Farey. Thus, the Farey Sequence is sometimes known as Haros-Farey Sequence. In the context of this paper, we use Farey Sequences to prove results relating to Diophantine Approximation. Farey Sequences are also closely related to continued fractions[2]. Their relationship is most clearly seen through the Stern-Brocot Tree[3], of which the left subtree containing the rational numbers between 0 and 1 is the Farey Tree. Most recently, continued fractions are being looked at in the context of p-adic numbers; a survey of recent work and open problems on the subject was written by Romeo[4]. As we'll also see later in the paper, continued fractions have applications in cryptography as well. Diophantine approximation[5], named after Diophantus of Alexandria, involves the approximation of real numbers using rational numbers. It is certainly not a topic limited to the scope of Farey Fractions. Most recently, James Maynard of Oxford University received a Fields Medal for his work in Diophantine Approximation, proving the Duffin-Schaeffer Conjecture[6], which is now aptly named the Koukoulopoulos–Maynard Theorem. Diophantine Approximation is also closely related to Transcendental Number Theory[6], and many of the techniques and results from Diophantine Approximation also apply to Transcendental Number Theory. Diophantine Approximation also have some applications in cryptography – for example[7], Worley's Theorem in Diophantine Approximation provides the basis for Wiener's Attack on the RSA cryptosystem. As we will also see, this is not the only theorem in Diophantine Approximation contributing to Wiener's Attack! In this paper we'll discuss and prove some interesting results that the Farey Fractions yield and their relationship to Diophantine Approximation. We'll also cover some nice geometric interpretations of the Farey Sequences. The assumed knowledge for the following paper is a thorough understanding of algebra and geometry.

Definitions and Properties

Definition 2.1. We define the Farey Sequence of order n, denoted F_n, as the sequence of rational numbers of the form $\frac{a}{b}$, where a and b are positive integers such that $1 \leq a \leq b \leq n$ and $\gcd(a,b) = 1$ (reduced form). We write the Farey Sequence in increasing order, starting from $\frac{0}{1}$ and ending with 1^1.

Example. The Farey Sequence of order 3 is the sequence of numbers

$$\left(\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right)$$

Due to the bounds on Farey fractions, we'll assume for the rest of the paper that fractions are in the range [0,1], unless otherwise specified, and that all fractions are in reduced form.

We can deduce one elementary property of the Farey Sequence.

Corollary 2.2. The number of elements in the Farey Sequence F_n is. $\sum_{i=1}^{n} \phi(n)) + 1$, where $\phi(n)$ is the Euler Totient Function, which counts the number of integers up to n relatively prime to.n.

Proof: The +1 accounts the fraction $\frac{0}{1}$ each element $\phi(i)$ of the summation accounts for the number of fractions in reduced form with denominator i in the interval $[0,1], \square$

More complicated questions arise when talking about the mechanics of the Farey Sequences. For example, given two rational numbers in the interval [0,1], are they ever consecutive in some Farey Sequence?

Theorem 2.3. Two distinct rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in F_b where $k = \max(\text{b,d})$, if and only if $ad - bc = \pm 1$.

We will prove this in a few parts. First, we introduce the notion of the mediant of two fractions.

Definition 2.4. The mediant of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is defined $as \frac{a+c}{b+d}$

The mediant of two fractions offers an interesting property if these two fractions happen to be consecutive in some Farey Sequence.

Proposition 2.5. $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

Proof: Observe that

$$\frac{a}{b} < \frac{a+c}{b+d} \Leftrightarrow ab + ad < ab + bc \Leftrightarrow ad < bc$$

which follows from $\frac{a}{b} < \frac{c}{d}$ On the other side,

$$\frac{a+c}{b+d} < \frac{c}{d} \Leftrightarrow ad + dc < bc + cd \Leftrightarrow ad < bc \square$$

in a similar fashion.

Corollary 2.6. For consecutive Farey Fractions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a+c}{b+d}$ is in reduced form.

Proof: Observe that $b + d \leq 2\max(b,d)$. Since $\frac{a+c}{b+d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$ in some Farey Sequence, the fraction $\frac{a+c}{b+d}$ in reduced form cannot have denominator less than or equal to $\max(b,d)$. Suppose for contradiction's sake that $\text{god}(a+c, b+d) = k > 1$. Then

$$\frac{a+c}{b+d} = \frac{\frac{a+c}{k}}{\frac{b+d}{k}}$$

But $\frac{b+d}{k} \leq \frac{b+d}{2} \leq \max(b,d)$, a contradiction to the consecutive nature of $\frac{a}{b}$ and $\frac{c}{d}$, hence $\frac{a+c}{b+d}$ is in reduced form. $\square$

The key idea is that each fraction is the mediant of two other consecutive fractions of a previous sequence. This can also be phrased, as:

This property is key to proving many identities about Farey Fractions. For example, now we can go ahead and prove Theorem 2.3. There is an algebraic proof using mediants, but it's. nicer to consider a geometric interpretation of the mediant of two consecutive Farey fractions; we present it below.

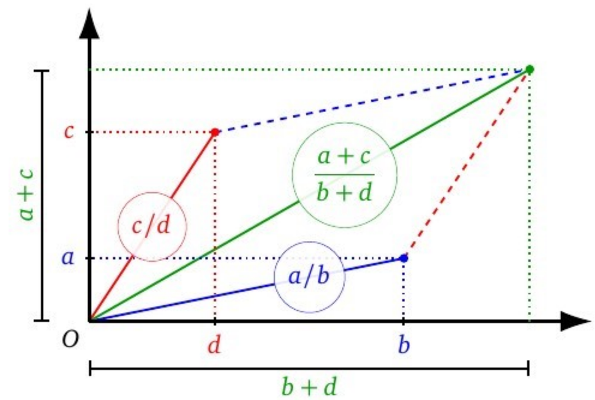

Figure 1: The mediant of two Farey Fractions [8]

Consider a model as in Figure 1 where two consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ can be represented as the vectors $u = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v = \begin{bmatrix} c \\ d \end{bmatrix}$. Then, Corollary 2.7 is represented by the fact that no lattice points are present inside the parallelogram formed by the vectors, as shown above. Note that if a lattice point (x_1, y_1) is inside the parallelogram, then there exists a fraction between $\frac{a}{b}$ and $\frac{c}{d}$, namely $\frac{x_1}{y_1}$; moreover, this point satisfies $y_1 < b + d$, implying that a fraction arises before $\frac{a+c}{b+d}$, a contradiction. Hence, no lattice points exist inside the parallelogram.

This gives rise to a proof of Theorem 2.3, which Amen[9] presents as follows. From Pick's Theorem, the area of the parallelogram with sides being the vectors u and v is equal to $I + \frac{B}{2} - 1$, where I is the number of interior points in the parallelogram and B is the number of boundary lattice points of the parallelogram. Note that $I = 0$ and $B = 4$ for consecutive Farey Fractions; $B = 4$ arises from the vertices of the parallelogram, and we've already seen that $l = 0$. Hence,

$$A = 0 + 4/2 - 1 = 1$$

At the same time, from the Shoelace Formula, the area of such a parallelogram is $|ad - bc|$, hence $|ad - bc| = 1$.

In summary:

Corollary 2.8. If no integer lattice points exist strictly inside the parallelogram generated by the vectors by the vectors $u = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v = \begin{bmatrix} c \\ d \end{bmatrix}$, where $a < b$ and $c < d$, then $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in the Farey Sequences $F_{\max(b,d)}$ up until $F_{\text{b}+d}$.

Diophantine Approximation

The central idea around this is approximating irrational numbers with rational numbers. From the denseness of the rational numbers, we know that, for every real number x and $\varepsilon > 0$, there exists integer p, q with

$$\left|\frac{p}{q} - x\right| < \varepsilon$$

As an example, we'll use

$$\sqrt{2} = 1.4142135\ldots$$

We could get approximations by taking fractions like

$$\frac{14142}{10000} = \frac{7071}{5000}$$

but this is pointless. Diophantine Approximation aims to minimize the denominator while getting as close as possible to the value at hand.

Theorem 3.1. (Weak Diophantine Approximation.) For every positive irrational number x and $\varepsilon > 0$, there exist infinitely many p, q with

$$\left|\frac{p}{q} - x\right| < \frac{1}{2q}$$

Proof: Bound x between any two rational numbers with the same denominator, ie. choose p_1, q_1 such that

$$\frac{p_1}{q_1} < x < \frac{p_1 + 1}{q_1}\square$$

Then choose the fraction x is closer to; since the sum of the distances from x to both fractions adds to $\frac{1}{q_1}$, it follows that either $\left|\frac{p_1}{q_1} - x\right| \leq \frac{1}{2q_1}$ or $\left|\frac{p_1+1}{q_1} - x\right| \leq \frac{1}{2q_1}$ In fact, this bound is tight because x is irrational.

The choice of p_1 and q_1 can be made for any positive integer q_1, hence infinitely many such appreximations exist.

This bound is a little weak, however, as evidenced by the fact that it can be done for all choices of q. If we take a look at the example of $\sqrt{2}$ again, then $\frac{99}{70}$ is a very tight approximation for $\sqrt{2}$; in particular,

$$\left|\sqrt{2} - \frac{99}{70}\right| = 0.00007215191 \approx 7 \cdot 10^{-5}$$

This looks more like an approximation of $\frac{1}{q^2}$ or $\frac{1}{2q^2}$ distance away; how close can we get?

In Theorem 3.1, we bounded x by two fractions of the same denominator. Can we find a better bound, though? The answer is through Farey Fractions.

Theorem 3.2. For every positive irrational number x and $\varepsilon > 0$, there exist infinitely many $p, q \in Z$ with $\left|\frac{p}{q} - x\right| < \frac{1}{2q^2}$'

Proof: Without loss of generality, assume $0 < x < 1$. (Otherwise, we can shift x to be in this interval.) Now, in some Farey Sequence F_n, bound x by the two consecutive fractions $\frac{a}{b}$ and $\frac{c}{a}$ with $\frac{a}{b} < \frac{c}{d}$ We claim that either

$$\left|\frac{a}{b} - x\right| < \frac{1}{2b^2}$$

$$\left|\frac{c}{d} - x\right| < \frac{1}{2d^2}$$

Suppose for contradiction that neither of these are true. Then we must have

$$\left|\frac{a}{b} - x\right| + \left|\frac{c}{d} - x\right| \geq \frac{1}{2b^2} + \frac{1}{2d^2}$$

At the same time, we also have

$$\left|\frac{a}{b} - \frac{c}{d}\right| = \left|\frac{ad - bc}{bd}\right| = \left|\frac{1}{bd}\right| = \frac{1}{bd}$$

following from Theorem 2.3. Now observe that

$$\left|\frac{a}{b} - x\right| + \left|\frac{c}{d} - x\right| = x - \frac{a}{b} + \frac{c}{d} - x = \frac{c}{d} - \frac{a}{b}$$

Thus, we must have

$$\frac{1}{2b^2} + \frac{1}{2d^2} \leq \frac{1}{bd} \Leftrightarrow 2bd \geq b^2 + d^2 \Leftrightarrow 0 \geq (b-d)^2\square$$

This is a contradiction as $b \neq d$; no two consecutive Farey Fractions apart from $\frac{0}{1}$ and $\frac{1}{1}$ have the same denominator. This follows from Theorem 2.3; if $b = d$, then $b(a-c) = 1$, which is impossible for integers a, b, c unless $b = 1$. Thus, at least one of the inequalities holds, implying that we've found an approximation.

In fact, as n grows larger, the fractions generated by $\frac{a}{b}$ and $\frac{c}{n}$, i.e. $\frac{a+c}{b+d}$ and so forth, will split the consecutive fraction interval in which x is in infinitely many times; thus, infinitely many such approximations exist for x.

As an interesting sidenote, the above theorem feeds into Legendre's Theorem on Continued Fractions, which is as follows:

Theorem 3.3. (Legendre's Theorem on Continued Fractions.) Consider a positive irrational number x and $p, q \in Z$ with $\left|\frac{p}{q} - x\right| < \frac{1}{2q^2}$. Then $\frac{p}{q}$ is a convergent of the continued fraction expansion of x.

Proof: See here[10]. $\square$

Theorem 3.3 has applications in public key cryptography. In particular, Wiener's Attack[11], a method of attacking the RSA cryptosystem, is based on this theorem. We now return to tour discussion of Theorem 3.2. Do there exist better constants c apart from 2 for $\frac{1}{cq^2}$? As it turns out, there do. The proof once again follows from Farey Fractions!

Theorem 3.4. (Hurwitz's Approximation Theorem) For every positive irrational number x , there exist infinitely many $p,q \in Z$ with

$$\left|\frac{p}{q}-x\right| < \frac{1}{\sqrt{5q^2}}$$

Proof: We'll follow an algebraic proof of Zukin's [12].

In particular, we'll show that the above inequality is satisfied by one of the fractions

$$\frac{a}{b}, \frac{c}{d}, \frac{a+c}{b+d}$$

where $\frac{a}{b}$ and $\frac{c}{\alpha}$ are consecutive in some Farey Sequence. Assume without loss of generality that x is bounded by $\frac{a+c}{b+d}$ and $\frac{c}{d}$, For contradiction purposes, assume that

$$x-\frac{a}{b} \geq \frac{1}{\sqrt{5}b^2}$$
$$x-\frac{a+c}{b+d} \geq \frac{1}{\sqrt{5}(b+d)^2}$$
$$\frac{c}{d}-x \geq \frac{1}{\sqrt{5}d^2}$$

For simplicity sake, let $a+c=e$ and $b+d=f$. Adding the first and third inequalities yields

$$\frac{c}{d}-\frac{a}{b} \geq \frac{1}{\sqrt{5}b^2}+\frac{1}{\sqrt{5}d^2}$$

and adding the second and third inequalities yields

$$\frac{c}{d}-\frac{e}{f} \geq \frac{1}{\sqrt{5}d^2}+\frac{1}{\sqrt{5}f^2}$$

Since $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in some Farey Sequence as is $\frac{c}{f}$ and $\frac{c}{d}$, we can write

$$\frac{1}{bd} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b^2}+\frac{1}{d^2}\right)$$

$$\frac{1}{df} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{d^2}+\frac{1}{f^2}\right)$$

Clearing the denominators gives us

$$bd\sqrt{5} \geq b^2+d^2$$
$$df\sqrt{5} \geq d^2+f^2$$

Now, adding the two yields

$$d\sqrt{5}(b+f)=d\sqrt{5}(2b+d) \geq b^2+2d^2+f^2=2b^2+3d^2+2bd$$

which can be rewritten as

$$0 \geq \frac{1}{2}((\sqrt{5}-1)d-2b)^2$$

Thus,

$$2b=d(1-\sqrt{5}) \Leftrightarrow \sqrt{5}=1-\frac{2b}{d}\square$$

a contradiction because $\sqrt{5}$ is irrational. Thus, at least one of the three original inequalities holds.

As we take fractions from more advanced Farey Sequences the fractions that bound x change, generating infinitely many approximations $\frac{p}{n}$, as desired. a

It turns out [10] that $\sqrt{5}$ is actually the maximum ε for which

$$\left|x-\frac{p}{q}\right| < \frac{1}{\varepsilon q^2}$$

has infinitely many rational solutions (p,q), for all irrational x. For $\varepsilon > \sqrt{5}$, the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ has only finitely many approximations. In fact, for $\varepsilon = \sqrt{5}$, all rational approximations for ϕ satisfying Theorem 3.4 fall under the family of rationals of the form

$$\frac{F_{\mathrm{n}+1}}{F_{\mathrm{n}}}$$

with F_n, F_{n+1} being the nth and $(n+1)$ th Fibonacci Numbers. Far $\xi > \sqrt{5}$, there exists $n_1 > 0$ such that, for all $n_2 > 0$, the fraction $\frac{F_{n_2+1}}{F_{n_z}}$ is not a valid approximation. It then follows that, før $\varepsilon > \sqrt{5}$, the golden ratio has finitely many approximations. For this reason, the golden ratio is known as one of the most difficult numbers to approximate.

What if we reverse this idea, and consider constants ε for a fixed x such that infinitely many $\frac{p}{q}$ exist?

Corollary 3.5. The number $\sqrt{2}$ has infinitely many rational approximations using, $\varepsilon = 2\sqrt{2} > \sqrt{5}$; in other words, infinitely many p,q with $\gcd(p,q)=1$ exist such that

$$\left|\sqrt{2}-\frac{p}{q}\right| < \frac{1}{2\sqrt{2}q^2}$$

Outline: Consider the Pell Equation

$$x^2-2y^2=1$$

As with all Pell Equations of the form $x^2-Dy^2=1$ with D being a non-square, this equation has infinitely many solutions. Specifically, we can take the solutions from the coefficients of $(3+2\sqrt{2})^k$. In other words, if

$$(3+2\sqrt{2})^k = a_k + b_k\sqrt{2}$$

then $(x,y)=(a_k,b_k)$ is a solution to the Pell Equation.

How can this be adapted to Corollary 3.5? The idea is that $\frac{a_i}{b_i}$ approximates $\sqrt{2}$ very well for all i. For example, using $i=1$, we have

$$\left|\sqrt{2}-\frac{3}{2}\right| \approx 0.08579, \frac{1}{8\sqrt{2}} \approx 0.08839$$

As i gets larger and larger, it follows that

$$\lim_{i\to\infty}\left|\frac{b_i\sqrt{2}-a_l}{\frac{1}{2b_i\sqrt{2}}}\right|=1$$

where the quotient keeps increasing as it reaches 1 . Rearranging then gives the result, a As a matter of fact, one can replace the $\sqrt{2}$ on the left-hand side with nearly every irrational number x, which brings us to our following (albeit brief) topic...

Lagrange Numbers

Consider the inequality

$$\left|x-\frac{p}{q}\right|<\frac{1}{cq^2}$$

for a constant c. We covered the cases of $c=\sqrt{5}$ and $c=2\sqrt{2}$. The natural question that arises is: why these two numbers?

Recall in the discussion after Theorem 3.4 that $\sqrt{5}$ is the maximum c such that

$$\left|x-\frac{p}{q}\right|<\frac{1}{cq^2}$$

has infinitely many approximations $\frac{p}{q}$. Moreover, we mentioned that, for $c>\sqrt{5}$, the golden ratio $\phi=\frac{1+\sqrt{5}}{2}$ cannot be approximated. What if we don't consider ϕ and it's associated irrationals (i.e. $\phi+c$ for a rational c)? Then, we can improve our constant c. In particular, infinitely many approximations for $\frac{p}{q}$ exist where

$$\left|x-\frac{p}{q}\right|<\frac{1}{2\sqrt{2}q^2}$$

where x is not one of ϕ 's associated rationals. So, in fact, we can generalize Corollary 3.4 to all numbers x.

Now, just as ϕ is approximated with difficulty for $c=\sqrt{5}$, the value $\sqrt{2}$ is approximated with difficulty using $2\sqrt{2}$. In other words, there do not exist infinitely many approximations $\frac{p}{q}$ with

$$\left|\sqrt{2}-\frac{p}{q}\right|<\frac{1}{cq^2}$$

for c $c>2\sqrt{2}$. Now, once again barring $\sqrt{2}$ and it's associated cationals, we can increase the constant c further.

These constants c, i.e. constants that can be used barring "bad"irrational numbers, are known as Lagrange Numbers, denoted L_n. Taking all these irrational numbers forms the Lagrange Spectrum. There's a curious relationship[13] with Lagrange Numbers and Markov Numbers that does not relate to Diophantine Approximation, but is nevertheless curious. Markov Numbers describe all positive integers x that are a solution to the equation

$$x^2+y^2+z^2=3xyz$$

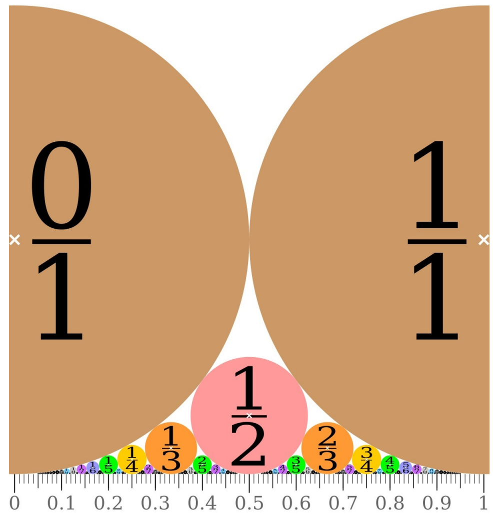

Figure 2: A picture is worth a thousand words. [1]

for positive integers duby, When rearranged in increasing order, it follows that

$$L_n=\sqrt{9-\frac{4}{M_n^2}}$$

For example, we have $M_1=1$ due to the triple($1,1,1$), so $L_1=\sqrt{5}$ as expected. Moreover, $M_2=2$ due to the triple $(2,1,1)$, so $L_2=\sqrt{8}=2\sqrt{2}$. Als, through this equation, we get the following obvious result:

Corollary 4.1. $\lim_{n\to\infty}L_n=3$.

Moreover, infinitely many Markov Numbers exist[14], due to an algorithm for generating a new triple given the triple (xyz) satisfying the Markov Diophantine Equation. Hence:

Corollary 4.2. Infinitely many Lagrange Numbers exist. In other words, we can keep increasing the constant c, which grows asymptotically towards 3 .

Ford Circles

When talking about Farey Sequences and Diophantine Approximations, it's impossible to not mention Ford Circles as well. The topics of Farey Sequences and Diophantine Approximation go hand in hand because Farey Sequences deal with fractions of a maximum denominator, while Diophantine Approximation is about finding the closest approximation to an irrational number given limits on the denominator. Ford Circles provide the perfect geometric representation of these two concepts.

Figure 2 shows the Ford Circles diagram. Reading left to right, a few Farey Sequences can be seen when looking at the fractions written in each circle. Why is this?

Construction[15]. For each $\frac{a}{b}$ in F_n, draw the circle centered at $\left(\frac{a}{b},\frac{1}{2b^2}\right)$ with radius $\frac{1}{2b^2}$. (Do this as n approaches ∞).

The main observation to be made from this picture is that two circles appear to be tangent if and only if their corresponding Farey Fractions are consecutive in some Farey Sequence.

This is the main result following from Ford Circles.

Theorem 5.1. Take two Farey Fractions $\frac{a}{b}$ and $\frac{c}{d}$ and consider the circles generated by the above construction. The circles are externally tangent if and only if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in some. F_n.

Proof: For two circles to be tangent to each other, the sum of their radii must be equal to the distance between their centers. In other words, we must have

$$\frac{1}{2b^2}+\frac{1}{2d^2}=\sqrt{\left(\frac{a}{b}-\frac{c}{d}\right)^2+\left(\frac{1}{2b^2}-\frac{1}{2d^2}\right)^2}$$

Squaring both sides and rearranging, this is equivalent to

$$\left(\frac{a}{b}-\frac{c}{d}\right)^2=\left(\frac{1}{2b^2}+\frac{1}{2d^2}\right)^2-\left(\frac{1}{2b^2}-\frac{1}{2d^2}\right)^2=\frac{1}{b^2d^2}$$

Thus, we must have

$$\left|\frac{a}{b}-\frac{c}{d}\right|=\frac{1}{bd}\Leftrightarrow|ad-bc|=1\square$$

which is Theorem 2.3. The theorem is bidirectional, so we reach our desired conclusion. □

How does this relate to Diophantine Approximation? Consider any irrational number ε and represent it as an x-coordinate. The idea is that there exists infinitely many Ford Circles that contain a point ε as their x-coordinate:

Corollary 5.2. The line $x=\varepsilon$ intersects infinitely many Ford Circles, for all irrational $0<\varepsilon<1$. Moreover, if $x=\varepsilon$ intersects a Ford Circle whose center has coordinate $\frac{a}{b}$, then $\frac{a}{b}$ approximates ε. Shifting ε by an integer yields the same result for all ε.

For example, let ε_1 be an irrational number with $0<\varepsilon_1<1$. The line $x=\varepsilon_1$ intersects one of the circles denoted by $\frac{0}{1}$ or $\frac{1}{1}$ in Figure 2. So, one of $\frac{0}{1}$ or $\frac{1}{1}$ approximates ε; in other words, one of the following is true:

$$\left|\frac{0}{1}-\varepsilon\right|<\frac{1}{2(1)^2}$$
$$\left|\frac{1}{1}-\varepsilon\right|<\frac{1}{2(1)^2}$$

This offers an alternate geometric interpretation for why 2 is a viable constant in the inequality

$$\left|x-\frac{p}{q}\right|<\frac{1}{2q^2}$$

for all irrational x.

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