Understanding Black Hole Under the Lens of the General Relativity

Lyujia Pan

Received September 14, 2023
Accepted June 25, 2024
Electronic access August 31, 2024

This paper focuses primarily on the interpretation, consequences, and different metrics of General Relativity, and how it can describe the spacetime near and inside of a black hole. Einstein came up with a general theory which incorporates special relativity by putting space and time into one single concept, and describing gravity as the curvature of spacetime. With the advent of this theory, ideas such as black holes appeared and have been studied since then. In this paper, we will briefly discuss some of the concepts of special relativity, and mainly focus on the meaning of each component of general relativity and different solutions for it and the prediction for black holes. The paper will include topics such as FLRW equations, Hawking radiation, the information paradox, and more. The general theory of relativity and quantum mechanics both reach their limitations at the singularity of the black hole; hence a theory which reconciles them is yet to be discovered.

Introduction

With the advent of the general theory of relativity, problems whose solutions eluded physicists for years, such as the orbit of Mercury, were solved. Mercury's orbit, unlike other planets in our solar system, rotates or precesses about 43 seconds of arch per century[1]. This weird phenomenon, however, turns out to be an inevitable reality described by general relativity as the mercury moves along the shortest path of the curvature of space caused by our Sun predicted by the theory. General relativity also predicts a phenomenon called gravitational lensing as a result of the curved geometry of space itself. This phenomenon allows an observer to see, for instance, a star blocked by another star at a different spot due to the curvature of the space caused by the large mass of the blocking star. This, coupled with other observable phenomena, ensured the accuracy of the general theory of relativity. General relativity opened a door for modern physics; diverse branches of discovery grew as physicists came up with different solutions to Einstein's equation. The FLRW metric[2], a solution which describes a spatially homogeneous and isotropic universe, led to the discovery of the expanding universe and the idea of dark energy (a type of energy that has an anti-gravity property that pushes the universe outward).Importantly, the Schwarzchild metric[3], solved by the German physicist Karl Shwarzchild[4], which describes the space geometry near a massive spherical object, predicted the existence of black holes[5].

The black hole, one of the most mysterious objects in our known universe, has unique properties such as the swapping of space and time coordinates inside the event horizon[6], the termination of space and time at the singularity[7], and more. It also creates problems for physicists such as the information paradox. This paper primarily focuses on the mathematical aspects of the general theory of relativity by decomposing and interpreting each symbol in Einstein's neat yet complex equation, and provides you with a comprehensive understanding of special and general relativity, the spacetime near and inside of a black hole, and some other branches of the discovery resulted from the general relativity equation[8].

Introduction to General Relativity:

After the publication of special relativity, the world did not feel satisfied with the theory as it was just too weird to be true. However, Einstein himself was also unsatisfied with special relativity as the theory only worked without acceleration. The equivalence principle, which states that the acceleration and gravitational force are indistinguishable and essentially the same thing (for example, if a person is in a room with no windows, he can not tell if he is on Earth or traveling in space if the room is accelerating at $9.81m/s^2$), serves as the foundation of General Relativity. The theory includes gravity and curved space as gravity-like effects could be switched on and off by motion, and the non-Euclidean[9] effects could be created by motion.

Gravity, according to general relativity, is not a force depicted in Newtonian physics; instead, it is the curvature of spacetime. An object without interference of any kind would follow a straight path; however, a massive object such as the Earth is near will distort the spacetime around itself and bend the straight lines of the space towards the center of the Earth. The object, following the path of space, would therefore move towards the center of the Earth too. This phenomenon is precisely what is called "gravity", but it is essentially the object following its shortest path in spacetime.

There are three categories of curvature: positive, zero (flat), and negative. A triangle on a positive curvature, like the surface of a sphere, the sum of the angles of the triangle, unlike in flat surfaces, would be greater than 180 degrees, whereas with negative curvature, the sum would be less than 180 degrees. Two parallel lines would never intersect on flat surfaces, however, they would if they are on the surface of a sphere, just like longitude and latitude of the Earth.

Notation of General Reletavity:

Before digging into the mathematical aspects of general relativity, understanding one of the fundamental equations of general relativity, the Einstein equation, is essential:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1}$$

In summary, the left-hand side of the equation describes how spacetime curves, and the right-hand side of the equation depicts the movement of mass in spacetime. We will now decompose the equation and interpret each symbol or notation in the equation.

A. The Christoffel Symbol

[10] In the cartesian coordinate system, the basis vectors form rectangular grids everywhere, and do not vary. In other words, the change of the basis vector with respect to the coordinate is zero. However, in curved space, the grids inside the coordinate system are not always regular, and the basis vector might vary as it moves along the coordinates. This change can be expressed as the derivative of the basis vector with respect to the coordinates as such:

$$\frac{d\vec{e}}{dx} \tag{2}$$

This additional vector depends solely based on the structure of the grid and can be expressed by its components, written as the capital letter gamma, and it contains different versions to represent the change of basis vectors going different directions:

$$\Gamma^{\gamma}_{\alpha\beta} \tag{3}$$

These numbers are called the Christoffel symbols, and they have the information about how the grid changes along each direction. The Christoffel symbol will later be used to derive the geodesic equation, which is applied to predict the trajectory of an object. In other words, it tells how an object is going to move in spacetime.

B. The metric Tensor

(g_{uv})[11] : On a curved spacetime, locations can be described using coordinates; however, without knowing the relationship, or more precisely, the distance between them, these points would be meaningless. One special case: the flat surface. The distance between two points on a flat surface where all the grids form identical regular squares, is obtained by the Pythagorean Theorem:

$$ds^2 = dx^0 + dx^1 \tag{4}$$

Where ds^2 is the length, and dx^0 and dx^1 are the two sides of the triangle. However, one of the limitations of the Pythagorean Theorem is that the two lines have to form an orthogonal system, that is, two lines form a 90-degree angle. In curved space, however, the grids are irregular; they usually do not have perpendicular axes like those in flat space. Therefore, a more general equation for calculating the distance between two points is expressed as the sum of all possible combinations of two sides (dx^0 and dx^1) and multiplied by some numbers in each term as:

$$dx^2 = Adx^0dx^0 + Bdx^0dx^1 + Cdx^1dx^0 + Ddx^1dx^1 \tag{5}$$

whereA,B, C, and D represent arbitrary numbers and directly depend on the shape of the grid. In our flat space case, the numbers would be A=1, B=C=0, and D=1, which shortens into the original Pythagorean equation. These numbers, usually represented in a grid with two indices, are called the metric tensor, and can be written as $g_{\mu\nu}$. The equation can be further generalize into such a form as:

$$dx^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{6}$$

The metric tensor $g_{\mu\nu}$ allows us to calculate the Christoffel Symbol described above; the change of the grid along each direction is directly related to the shaped of the grid just as the $g_{\mu\nu}$ indicates.

C. Riemann Curvature Tensor

[11]: On a flat sheet of paper, a vector will always point toward the same direction no matter how one translates it. However, the order of translation of the vector matters on a curved surface. Different orders of translation would result in vectors which are not equivalent to the initial vector. This difference of these vectors can be denoted by a new vector $\vec{R}$, where

$$\vec{R} = \frac{d}{dx^{\mu}}\frac{d}{dx^{\nu}}\vec{e}_{\beta} - \frac{d}{dx^{\nu}}\frac{d}{dx^{\mu}}\vec{e}_{\beta} \tag{7}$$

The vector $\vec{R}$ would be zero if the surface is flat, as the two vectors obtained from different translations will be identical; as the curvature gets bigger, the value of $\vec{R}$ will get larger as the difference increases. Here, note that this concept appears to be similar to the Christoffel Symbol, but they are not equivalent. The Christoffel symbol describes how the grid changes as the derivatives of basis vectors with respect to coordinates, and is a tool to calculate the Reimann curvature tensor, which shows every possible way basis vectors can change when translated.

Since $\frac{d}{dx^{\nu}}\vec{e}_{\beta}$ and $\frac{d}{dx^{\mu}}\vec{e}_{\beta}$ the derivatives of the basis vector with respective to coordinates, give us Christoffel symbols, the equation can be finalized after a few replacements:

$$R^{\alpha}_{\beta\mu\nu} = \frac{d\Gamma^{\alpha}_{\beta\nu}}{dx^{\mu}} + \Gamma^{\lambda}_{\beta\nu}\Gamma^{\alpha}_{\lambda\mu} - \frac{d\Gamma^{\alpha}_{\beta\mu}}{dx^{\nu}} + \Gamma^{\lambda}_{\beta\nu}\Gamma^{\alpha}_{\lambda\mu} \quad (8)$$

The Reimann curvature tensor can fully describe the geometry of space curvature; however, it is extremely difficult to compute and some assistance is needed: the Ricci tensor and Ricci scalar.

D. Ricci Tensor[11]

: The Ricci Tensor essentially describes how volume changes when it moves along its curvature, and is represented as one row and one column for each coordinate. Each component in the Ricci tensor is calculated using the Reimann curvature tensor represented in Einstein Notation as such:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \quad (9)$$

Each component of the Ricci Tensor describes the change in volume in different directions.

E. Ricci Scalar[11]

: The Ricci scalar is a quantity R that describes the average change of the volume, and is calculated as such:

$$R = g^{\mu\nu}R_{\mu\nu} \quad (10)$$

On the surface of a sphere, the area between two parallel geodesics would get smaller as we move them towards each other. Here the Ricci scalar is easy to galculate as the Ricci tensor would be the same no matter what way we move these two geodesics as the surface of a sphere is symmetrical and variation of the volume would be identical in all directions. In general, the metric tensor allows us to find the Churistoffel symbol as we have to understand how the grid changes and how the basis vector changes along each direction. The Chatistoffel in turn cap be used to calculate the Riemann curvature tensor, which can fully describe the curvature of space. However, the Ricci tensor and Ricci scalar are assisting in finding the Riemann curvature tensor. Take a flat or Minkoxiki qpacetime for example, where no matter and hence no curvature exists. Here the derivative of basis vectors with respect to coordinates ($\frac{d\vec{e}}{dx}$) will be zero as they are the same everywhere in this flat spacetime, which will result in zero in the Churistoffel symbol, and in turn result in zero for the Rejmang curvature tensor. In this case, the Ricci scalar is zero too. It makes perfect genge as there is no curvature in flat space.

On a sphere, however, the calculation will be slightly difficult, as $\frac{d\vec{e}}{dx}$ is no longer zero. However, equation for the Ricci Scalar canstillbeforued after a geries of computations:

	t	x
t	T_{tt}	T_{tx}
x	T_{xt}	T_{xx}

$$R = \frac{2}{r^2} \quad (11)$$

Where r is the radius of the sphere. This result is also intuitive: as the radius of the sphere gets bigger, the curvature of its sphere gets gmaller (that is why people in the past thought the Earth was flat). As the radius approaches infinity, there is no difference from a flat surface, as R is approaching zero.

F. Energy-Momentum Tensor[11]

The equation for General Relativity includes both the curvature of gaceftime and the movement of mass; we have now finished everything that accounts for space and ate left with the last tensor: the energy-momentum tensor (also known as the stress-energy tensor). For simplicity, consider only one spatial dimension and one time dimension. In this way, the tensor will only have four components that need to becsensidered. | T_{tt} shows how much energy is moving through time, which is also known as the energy density; T_{xt} and T_{tx} the same (both represent the momentum density, or equivalently, energy flux); T_{xx} shows the tendency energy has to push in one direction, also known as the pressure or momentum flux. In reality, however, there are three spatial dimensions which makes the tensor far more complex by including components such as viscosity and so on. The energy-momentum tensor essentially describes how much energy or mass (equivalent via Einstein's , $\square = \square\square^2$) moves along with space and time coordinates. In other words, it shows the movement of mass in the universe.

G. Geodesic equation[12]

: If the velocity of an object moving in space is given, one can predict its trajectory by plotting the velocity along its direction of movement. Without mass nearby, the object tends to move in a straight line. Every object in the universe moves in such a traiectorxwbich is called the "geodesic the shortest path an object can take. If an object moves on its geodesic without mass nearby, the geodesic is essentially a straight as there is no curvature presents; hence the acceleration would be:

$$\vec{a} = \frac{d\vec{v}}{d\tau} = 0 \quad (12)$$

Where $\vec{v}$ is the velocity and τ_{is} the proper time. For all vectors, in this case, velocity canben represented as the dot product of the components in each direction with the basis vector. Using Einstein notation, it is requresented, as such:

$$\vec{v} = v^{\alpha}\vec{e}_{\alpha} \tag{13}$$

Replacing the V with this equation, we get:

$$\frac{d\left(v^{\alpha}\overrightarrow{e_{\alpha}}\right)}{d\tau} = 0 \tag{14}$$

Which can be further expanded into:

$$V^{\alpha}\frac{d\vec{e}_{\alpha}}{d\tau} + \frac{dv^{\alpha}}{d\tau}\vec{e}_{\alpha} = \vec{0} \tag{15}$$

Using the Chrigtoffel symbol as discussed above, which can derived from $\frac{d\vec{e}_{a}}{d\tau}$, and after few replacements, we obtain:

$$\frac{dv^{\alpha}}{d\tau} = -\Gamma^{\alpha}_{\mu\nu}v^{\mu}v^{\nu} \tag{16}$$

This equation iscalled the geodesic equation. It allows us to calculate, as the left hand side of the equation indicates, the derivative of each component of the velocity of an object as proper time pasges. Therefore, using the geodesic equation, the trajectory of an object can be predicted if its velocity at a certain moment in time is provided.

Links of every Symbol in the Equation:

The geodesic equation, which describes the movement of an object based on the curvature of the space, by using the metric tensor, the Cfroatoffel symbol, the Riemann curvature tensor, the Ricci tensor, and the Ricci scalar; the Energy-Momentum tensor, which describes the presence of mass contained in the space. The equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{17}$$

Essentially links the curvature of space and the energy together, where the Riemann curvature tensor and metric tensor are on the left hand side and the Energy-Momentum tensor is on the right with some constants in them. The term

$$\frac{8\pi G}{c^2} \tag{18}$$

Contains the gravitational constant G in order to incorporate the Newtonian gravity as it still holds true when the gravity is weak. However, what does it mean by calculating the solution of the equation after all? By finding a solution, it means to find the geometry of the space under a certain context; it could be in a vacuum, outside a massive star, or even a black hole. Essentially, the mission is to find the representation for the metric tensor $g_{\mu\nu}$.

Example Of the Solution For General Relativity Equation:

In a lot of circumstances, knowing the geometry of spacetime outside massive spherical gbjects, such as planets and stars, can be helpful. This metric tensor. $g_{\mu\nu}$, found by the German physicist Karl Schwaszchild, is known as the Schwarzchild metric. This metric predicts the existence of an astronomical object with gravitational force that nothing can escape from, also known as the "black hole", which we will discuss in this section. The Schwarzchild metric is often represented in polar coordinates in such form:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{r_4}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

This metric describes the spacetime near an spherically symmetric, uncharged, and non-rotating, under the condition of a static object that is spacetime(meaning the coordinates of the metric does not depend on time). Using this metric, the equation for proper distance and proper time can bes obtained:

$$ds^2 = -c^2dr^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{1-\frac{r}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{19}$$

Where r_s is the Schwabychild radius[13] and:

$$r_s = \frac{2GM}{c^2} \tag{20}$$

and τ is proper time, t is the coordinate time, and r is the radial coordinate, which along with θ and φ, all describe the position of the observer. If there is no mass present or when $M = 0$, the Schwarzchild metric turms into the Minkowski metric[14] which describes the geometry of the spacetime in a vacuum (flat spacetime):

$$g = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

(figure.1)

The equation for proper distance and time would then become:

$$d\tau^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{21}$$

From calculating the proper time of the Schwwarzchild metric, we can find that there exists time dilation just like in Special Relativity. If r , the position of an observer is very far away from this massive planet, the term $1-\frac{r_s}{r}$ will be l_0 essentially not affecting the proper time. However, if the observer is close to this massive planet, as the value of r becomes closer and closer to the value of τ_s, this term is no longer 1 and will affect the proper time of the observer. In other words, as the observer gets closer to the planet, time would run slower compared to another observer far away.

Black Hole:

A star, like our sun, is at an equilibrium position where the fusion inside the star creates the forces, which counterbalances the gravitational force created by the mass of the star. However, the elements inside a star would oned dax burn outs the force created by the fusion therefore will not be strong enough to push against the gravitational force. In this case, the star collapses and forms a supemova. Depending on the mass of the star, it could produce two possible outcomes: a black hole would form if the mass of the star is large enough; otherwise, a neutron star would form.

The Schwarschild Metric And Event Horizon:

In the Schwaszochild metric $r_s s$, the Scshwarachild radius, is the radius that any object needs to be in order to become a black hole. r_s bas the speed of light in the denominator, which suggests that smaller objects such as our Earth would have to be compacted into a sphere with extremely small radius. In fact, the Schwarzchild radius is.also known as the event horizon of a black hole. Depending on its mass, the event horizon is a fixed distance of a black hole where nothing inside this range is able to escape, including light. Near but outside the event horizon, there is a region called the photo sphere, where light orbits around the black hole and is not forced to go into it nor to escape it. In other words, if a person can somehow reach the photo sphere, one can see the back of his head as the light of the back would reach the observer's eyes after orbiting around the black hole.

The Singularity Problem:

There seems to be a limitation of the Scschwazazzchild, metric. In the term $-\left(1-\frac{r_s}{r}\right)$ and $\frac{1}{\left(1-\frac{r_r}{r}\right)}$ there will be a problem if r $=$ 0 and $r = t_s$ respectively, as it does not make sense when the denominator of a fraction is zero. This problem is.salled the singularity problem[7]. In turns out the case where $r = r_s$ is a removable singularity, meaning that this singularity does not physically exist, it is merely a flaw in our choice of polar coordinates system. For instance, the point at the center is not defined in polar coordinates. When $r = 0$, the value of θ is meaningless as it does not provide a specific value that represents the center of the coordinate system. However, not being able to represent the center of the coordinate system does not mean the center does not exist, as it can, be perfectly, defined if another coordinate system such as the pot be removed. Therefore, it is a true singularity at the center of a black hole, where time and space literally come to a stop and general relativity no longer applies. There is no observational evidence to support or to study the singularity as no information can escape from the black hole. The singularity is like a hole in spacetime, where for a calculus based theory like general relativity, the singularity is indifferentiable hence the meaning of it can not be determined

A. Trapped surface and the singularity theorem:

Although a singularity cannot be observed directly, there are theories which made its existence certain. The weak energy condition in general relativity suggests that the gravitational field has a focusing property; that is, its geodesics tend to converge. In other words, two parallel straight lines will meet at one point as they follow the geodesic. Since the event horizon can be at different positions based on different observers, Roger Penrose[15] came up with a more definite idea: trapped surface[16]. The trapped surface is a region where, like the event horizon, all the geodesics inside are pointed towards the center of the black hole. For conceptual understanding, imagine two light rays being shot from the same point inside the trapped surface but at different directions. Based on the weak energy condition and the converging geodesics, the light rays must focus at the center of the black hole and crometo.and, end. It will be impossible for either of the light rays to continue going after the two cormyerge as there will be another shortest path, meaning another geodesic if that happens. It would be like traveling south and reaching the south poles, after which it is meaningless to go even more south. Here, as the two light rays meet at the center of the black hole, time and space cometroanond instead of goutherpests. This result is called "geodesic incompleteness"[17], where the smooth garderics. which can beedescribed bes calculus appear to have holes in them: singularities. Stephen Hawking[18] took this idea and implemented it into the expansion of the universe. He claimed that since everything inside the trapped surface including spacetime. itself has to converge at a point and comets.anand, the Big Bang Theory can be viewed the same way. The Universe started from a point, it then exploded and created the universe. However, this process cankex exatraed. Just like the trapped surface, everything therefore must cometo.an. end at the very beginning of the universe, including space and time. Therefore, it is meaningless to ask what was there before the big bang as there is no time and space before that. Hawking and Penrose further developed a complete theory of geodesic incompleteness called the "Penrose- Hawking Singularity Theorem"[17].

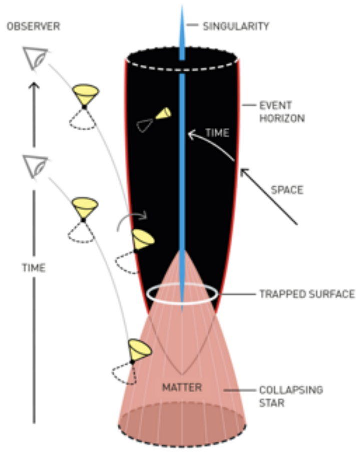

Figure:2

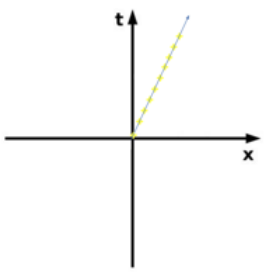

Figure:3[21]

Inside The Event Horizon/Trapped Surface:

The weirdness of the inside of a black hole is not just limited to its singularity; in fact, time and space swap when passing the event horizon or the trapped surface. To understand this, the concept of "light cone"[19] is indispensable.

A. Light Cone :

The German-Polish Mathematician Hermann Minkowski[20] coined the term "spacetime" based on the special theory of relativity, combining space and time, two seemingly unrelated concepts into one diagram.

The reality contains three spatial dimensions; however, for simplicity and the understanding of the diagram, only one special dimension will be used. Putting the time component on the vertical axis and the spatial component on the horizontal axis, we get a diagram which is called a Minkowski or spacetime diagram as such:

In order to let those two axes have the same units, time can be depicted as a length by measuring how far the light travels in a certain amount of time, which can be labeled as "ct" (c as

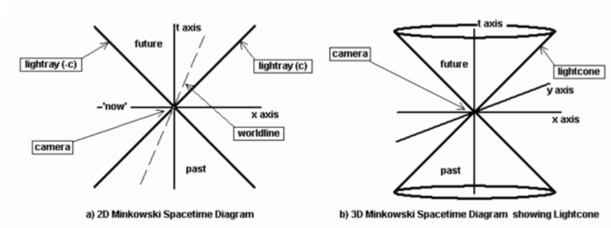

Figure:4[22]

the speed of light). Note that many physicists use natural units, where the speed of light (c) is equal to 1, and time and space are considered to have the same units.

Imagine living in a two dimensional world, a candle emits a circle of light which will surround an observer completely, and it is impossible to get out as nothing travels faster than the speed of light. As the circle gets bigger and bigger, the motion can be decomposed like movie frames and stacked image by image from the beginning, which forms a cone that grows bigger in time that is called the "light cone". A light cone restricts the cause and effect relationship in our observable universe and is an indispensable concept in both special and general relativity. Oftentimes there is a cone that flipped around representing the past on the diagram. Putting the lightcone onto the spacetime diagram, it would look as such:

The trajectories of light rays would be 45 degrees as we set $x = ct$, meaning the slope of the line would increase at the same rate for both coordinates. In other words, in one year, the light will always travel one light year.

The positive t direction represents the future, and the negative t direction represents the past. The cone which expands at the speed of light always orient itself in the direction of the time axis.

Everything in the diagram has a worldline, which is a curved trajectory representing an object's existence based on position and time. A person at rest, for example, has a vertical worldline as he moves forward in time at a fixed position. Worldlines inside of the cone are impossible to get out (as nothing travels faster than the speed of light), and are called "timelike", and obey the cause and effect relationship. For instance, one would not know anything about an explosion far away until his worldline meets the lightcone emitted by the explosion. Before that, the person would not experience this event. The worldlines outside of the cone are called "spacelike", which is physically impossible to reach as it has a smaller slope than the light ray, meaning that the speed would be faster than the speed of light.

B. How time and space swap inside the event horizon:

Light cones, however, do not always orient themselves in one direction. Where large mass, for instance, the sun, curves the spacetime, the light cones tend to also orient themselves towards the sun. One experiences this as “gravity”.

Similarly, if an observer approaches a black hole, his light cone would also bend towards the black hole. Nevertheless, an interesting phenomenon happens as one reaches the event horizon. As illustrated from the figure above, the event horizon itself serves as the light cone where the edge of the event horizon aligns with the light rays of the light cone. It would make intuitive sense as the light rays of the light cone restrain us from going backwards in time, which obeys the fact that anything reaching the edge or the inside of the event horizon can not escape from it. After one passes through the edge of the event horizon, the light cone of oneself would orient or the time axis would point directly towards the singularity by following the geodesics inside the black hole. As this happens, the singularity of the black hole is no longer correlated with the space axis, and is essentially in the future of the timeline. In other words, the singularity as one passes through the event horizon is no longer a place in space but an event in the future that is yet to happen; the event horizon on other hand, is an event in the past toward which one can not go back. Therefore, in order to truly understand the singularity, one has to physically go in there.

Hawking Radiation[23]

: Black holes, like any other astronomical objects, do not last forever. Stephen Hawking discovered that they would evaporate as he tried to combine quantum mechanics with general relativity.

According to quantum mechanics[24], a vacuum space is not really a vacuum; there are constantly virtual particles[25] being created near the black hole due to the Heisenberg Uncertainty Principle[26]. The Originally, the Heisenberg Uncertainty Principle states that:

$$(\delta x)(\delta p) \geq \frac{h}{4\pi} \tag{22}$$

where x and p are the position and momentum of a particle respectively. It states that the more accurately one detects a particle’s position, such as an electron, the less accurately one can measure its momentum, and vice versa. However thereis.anothersetsf.complementary warjableato which this. concept applies:

$$(\delta E)(\delta t) \geq \frac{h}{4\pi} \tag{23}$$

It suggests that time and energy work the same way. Essentially, pairs of virtual particles or waves can be created from vacuum and annihilate each other in a very short amount of time. This phenomenon is called the "quantum fluctuation"[27].

Imagine two people, one person is free falling into the black hole and the other stays still (or "motionless") at the edge of the event horizon in a spaceship accelerating towards the opposite direction preventing from falling. For the person free falling, he will perceive the space as "empty"as locally the spacetime is flat so that the positive and negative waves cancel each other out. However, things work differently for the person in the spaceship. Due to the acceleration of the ship, the person will perceive the waves with distorted frequencies - the waves have higher frequencies in the front of the spaceship than the back. Hence, for the person in the spaceship, the waves do not cancel each other out, meaning that he does not perceive an empty space but a space full of particles. Therefore, the existence of particles at the edge of the event horizon is relative and depends on the motion of the observers. However, as we move away from the black hole where the spacetime is flat, the particles that exist in a relative form become "real". This phenomenon is the Hawking radiation, where the virtual particles near the horizon become real particles as we move further away, and creation of the real particles must come from the mass of the black hole itself.

A more intuitive way to understand this is that when quantum fluctuations occur near a black hole, the negative virtual particles can be absorbed, and since time and space swap within a black hole as the positive particles escape and serve as the radiation. Over time, the black hole loses its energy and becomes smaller, eventually evaporating.

Interestingly, Hawking discovered that the radiation of the black hole matches the spectrum that an object would emit due to its temperature. The temperature equation[28] derived by Hawking depends on the mass of the black hole:

$$T = \frac{hc^3}{8_B M} \tag{24}$$

It is so far the best attempt that exists of reconciling quantum machines and general relativity. h here is the reduced Planck constant, and is used in quantum mechanics; c and G are the speed of light and gravitational constant, respectively, which are used in general relativity. $K_{B\,\text{is}}$ the Boltzmann constant in thermodynamics and M is the mass of the black hole. Normally, an object loses heat or energy as it radiates, however, black hole heats up as it radiates and becomes smaller. We can see the phenomenon through equation 24 . As the masu(M) of the black hole decreases, meaning getting smaller, the temperature increases. In other words, a large black hole has lower temperature than relatively smaller ones, and evaporates more slowly as they absorb negative virtual particles at a slower rate due to the gentle curvature of apacetime caused by the event horizons.

A. Information paradox:

As a black hole evaporates, it leaves behind nothing, making it impossible for us to trace back to its original state. That also indicates a problem - "the information paradox"[29] - as everything the black hole absorbs during its lifetime will be gone forever, which violates the law of conservation of energy/information. Some physicists believe that the information is spit back out into our universe, others, including Stephen Hawking himself in the beginning, believe that the information is forever lost.

One of the earliest proposed solutions to the Information paradox is the Einstein-Cartan theory. This theory suggests that the information taken by the black hole is transferred to another universe, which we have no access to, via a wormhole. Essentially, the information is still there even though we cannot detect it. However, from the discussion earlier, we know that from the perspective of an outside observer, the falling information is frozen in time due to the time dilation. Therefore, it is believed that the information falling into the black hole is still there, and is stored around the event horizon - no other universe needed.

However, this idea leads to another problem. Even though from the perspective of an outside observer, the information is "frozen in time", from the perspective of the falling information however, the information itself indeed falls into the black hole. It seems like a duplication of the information happens, which violates the law of conservation of the information. Though some physicists argue that the two copies of the information, like the position and the momentum according to the Heisenberg Uncertainty Principle, cannot be observed by one observer at the same time, hence the duplication technically can happen without violating anything.

Still, the mechanism behind this is still a mystery, and there is no evidence nor mathematical proof to indicate any of the theories is correct.

Future Research Directions

Information paradox is an unsolved problem, but merely one of the many. The future research directions of the black hole is broad. Currently, physicists are putting black holes as one of the candidates for the dark matter[30]; using black holes as a way to study quantum gravity[31]; using black holes to further investigate thermodynamics and entropy[32] .

Conclusion

Gravity, according to the general theory of relativity, is the curvature of spacetime. Objects in the universe tend to follow the geodesics of spacetime, which results in the phenomenon in which one object is attracted to another like the Sun and the Earth. One of the solutions of Einstein's equation is the Schwarzschild solution, which predicts the existence of the black hole. The singularity in the center of a black hole is like a hole in spacetime, making general relativity inapplicable as it incorporates calculus to describe the smooth geodesics in space. While general relativity successfully explains the grand scheme of the universe at the same time incorporates the Newtonian mechanics, it does not reconcile with quantum mechanics, which explains the reality at an atomic level, at the singularity in the black hole as they should be; perhaps there is something wrong with our current understanding of the singularity, or the way that gravity works on a quantum scale. With all this confusion, physicists have set out to discover a theory of quantum gravity which can solve all these problems: a theory of everything[33].

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