

# ON THE ZEROS OF THE RIEMANN ZETA FUNCTION

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This paper provides an introduction to the theory of complex analysis and applies it to prove various fundamental properties of the Riemann zeta function. Specifically, integral identities for the zeta function are derived, its zeros are analysed both on and off the critical strip, and with the aid of a python script, some nontrivial zeros are approximated. The first section of the paper discusses the necessary background knowledge required to understand the findings in the paper. It covers holomorphic functions and Cauchy's theorem regarding contour integrals. It also provides some examples for each topic. Section 3 discusses the zeta function and the gamma function, and it states a relationship between the zeta and gamma functions. This section establishes a base to study the zeros of the zeta function. Section 4 is on the zeros of the zeta function, and provides proof that the zeta function has no zeros in some regions, some trivial zeros, and the non-trivial zeros on the critical strip. Section 5 explains how the author wrote a program to compute the zeros of the zeta function on the critical line. The data collected in this paper will add to the list of the roots of the Reimann zeta function, and the program provides a new way of empirically computing the zeros. The plot of the function may provide additional information about the function, and questions such as 'why is the shape of the plot a cardioid?' can be asked to investigate further the zeta function.

## Introduction

The zeta function was first studied by Leonhard Euler. He was interested in its properties as a function that could be used to calculate the sums of infinite series. Euler discovered that the zeta function could be used to calculate the sum of the series  $1 + 2 + 3 + 4 + \dots$ , which is equal to  $-1/12$ . He also showed that the zeta function has a simple pole at 1 with residue<sup>1</sup>.

Euler's work on the zeta function lay dormant for many years until it was rediscovered by Bernhard Riemann in the 19th century. Riemann was interested in the zeta function as a way to understand the distribution of prime numbers. He showed that the zeta function has an infinite number of zeros, all of which lie on the line  $1/2 + it$ . He also showed that the zeta function is connected to the prime number theorem, which states that the density of prime numbers is  $\frac{n}{\ln(n)}$ .

The zeta function has since been studied by many mathematicians and has found applications in a variety of fields, including physics, finance, and cryptography. The zeta function has many interesting properties, and the zeros of the zeta function are especially interesting. The zeta function has a close connection to the prime numbers and the zeros of the zeta function can give us information about the distribution of the prime numbers. Euler's work indicated a possible connection of the zeta function to the prime numbers. Riemann then discovered that the roots, of the zeta function, are what connect the function to the prime numbers. He hypothesized that the only zeros of the zeta function inside the critical strip  $0 < \Re(s) < 1$  would lie on the critical line, or the line where

$$\Re(s) = \frac{1}{2}.$$

This paper portrays the author's elementary understanding of complex analysis and that of the zeta function. The paper presents a list of the zeros of the zeta function that the author generated by writing a program. Employing the zeta function, we write a program that generates an array of the list of zeros of the zeta function. A final evaluation of the list revealed that the zeros found were accurate when compared to what other mathematicians have found. The first section of the paper is the Introduction, which displays the paper's findings and results. The second section very briefly goes over some concepts in complex analysis. It covers some of the prerequisites needed to read and understand the paper. Section 2.1 discusses holomorphic functions. A holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighbourhood of every point in its domain. An alternate procedure is to use the Cauchy-Reimann Equations, which are not mentioned in the paper but are explained in full detail in the book *Complex Analysis*<sup>2</sup>. The section then includes some examples of functions that can and cannot be considered holomorphic. Additional examples are given in<sup>2</sup>. The section then discusses Cauchy's Theorem and contour integration. Section 3 provides a proof that if  $\Re(s) > 1$  then

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{u^s - 1}{e^u - 1} du.$$

This equation establishes a relation between the zeta function and the gamma function. The paper states that these two func-

tions are very deeply interrelated and the study of the gamma function is crucial in the study of the zeta function and that of the prime numbers. Section 4 discusses the zeros of the zeta function; it first proves that if  $\Re(s) > 1$  then  $\zeta(s)$  converges. The proof uses Cauchy's Integral theorem on the absolute value of the zeta function. Then, a check for convergence proves this result. Then the proof provides that if  $\Re(z) > 1$  then  $\Gamma(z) > 1$  has no zeros. In other words,  $\Re(z) > 1$  is a zero-free region of the gamma function. This is proven by a contradiction, where it is assumed that a zero of the function exists on this region, but is contradicted by the isolated zeros property of holomorphic functions. We then prove that  $\Re(s) > 1$  is a zero-free region for  $\zeta$ . This is done by writing  $\zeta$  as an exponential in terms of the Lambda function,  $\Lambda(n) = \ln(p)$  if  $n = p^m$  for some prime  $p$  and 0 otherwise, as

$$\zeta(s) = \exp\left(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln(n) \cdot n^s}\right)$$

The proof uses properties of the zeta function and its zeros to simplify this expression, and by using a trigonometric identity it is proven that  $\zeta$  has no zeros for  $\Re(z) > 1$ . The paper continues with another crucial result in the study of the zeta function, that is, the only zeros of the zeta function for  $\Re(s) < 0$  are the negative even integers. This proof is carried out using the following functional Equation for the  $\zeta$ -function.

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s)$$

The zeros of this function will occur at the poles of  $\Gamma(\frac{s}{2})$ , which are at the negative even integers. Hence, it proves that the only zeros  $\zeta$  for  $\Re(s) < 0$  are at the negative even integers, with imaginary part zero.

Section 5 comprises the contribution of this paper. This section presents a program the author wrote to compute the zeros of the zeta function. The program works by computing  $\zeta(s)$  for  $\Re(s) = \frac{1}{2}$ . It then makes a list of all the outputs, and if the absolute value of the output is close to zero, then the value is counted as a zero. The program computes values of  $s$  for  $\Im(s) \in [0, 1000]$ . There are a few uncertainties in the program, which are discussed briefly in the paper.

## Complex Analysis Background

This section discusses the necessary background information needed to understand and comprehend the results of this paper. It provides the knowledge required for holomorphic functions and Cauchy's Theorem, which pertain to contour integration. These are crucial to study the zeta function. These topics are explained in much more detail in<sup>2</sup>.

## Holomorphic functions

To understand Cauchy's theorem, it is essential to understand what holomorphic functions are, since they are a pillar of Cauchy's Theorem. A function  $f$  is said to be holomorphic at a point  $z_0$  in an open set  $\Omega$  in  $\mathbb{C}$ , if the following limit converges

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

where  $h \in \mathbb{C}, h \neq 0, z_0 \in \Omega$  with  $z_0 + h \in \Omega$ . A point to note is that because  $h \in \mathbb{C}$  it may approach 0 from any direction, unlike on  $\mathbb{R}$  where there are only two possible directions. The above equation is for a point  $z_0$  on the function  $f$ , which means that if the limit exists then the function is said to be holomorphic at the point  $z_0$  and not for all  $z \in f$ . A function  $f$  is said to be holomorphic if

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

exists on every point on the function  $f$ . The above limit is the derivative of a function which means that it can be expressed as

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

This implies that for an open set  $\Omega$  on  $\mathbb{C}$  and a complex-valued function  $f$  on  $\Omega$ ,  $f$  is said to be a holomorphic function if the derivative exists on every point of the function  $f$ . This also means that the function is continuous and an intuitive way of defining a holomorphic function is that if a complex function  $f$  is smooth, then  $f$  is a holomorphic function.

### Few Examples

**Example 1.** The function  $f(z) = z^n$  is holomorphic for  $n \in \mathbb{N}$   
Let  $F(z) = z^n$

$$\begin{aligned} f'(z) &= \lim_{h \rightarrow 0} \frac{f(z+h)^n - z^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{z^n + nz^{n-1}h + \binom{n}{2}z^{n-2}h^2 + \dots + h^n - z^n}{h} \\ &= \lim_{h \rightarrow 0} \left( nz^{n-1} + \binom{n}{2}z^{n-2}h + \binom{n}{3}z^{n-3}h^2 + \dots + h^{n-1} \right) \\ &= nz^{n-1} \end{aligned}$$

Since the limit exists  $f$  is holomorphic. Proof. Let  $f(z) = \underline{z}$

$$\begin{aligned} f'(z) &= \lim_{h \rightarrow 0} \frac{\overline{z+h} - \bar{z}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\bar{h}}{h} \end{aligned}$$

## Cauchy's Theorem and Its applications

This section will focus on Cauchy's Theorem and Cauchy's Integral formula, which are very important in the field of complex analysis and line integrals in the complex plane. Cauchy's theorem states that if  $f$  is a holomorphic function in an open set  $\Omega$  and  $\gamma$  is a closed curve (meaning that it has the same start and end point) in  $\mathbb{C}$  such that its interior is also contained in  $\Omega$  then

$$\oint_{\gamma} f(z)dz = 0$$

The above equation is very loosely stated because  $\Gamma$  is a separate variable from the complex numbers. This problem may be proven straightforwardly by parametrized  $\gamma$  and  $f(z)$  so the integral is expressed in terms of a single variable: Define  $z : [a,b] \rightarrow \mathbb{C}$

$$\oint_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt = 0$$

$$z(t) = x(t) + iy(t)$$

Now that the integral is in terms of one variable it becomes relatively easy to evaluate. Even a basic proof of Cauchy's Theorem requires a fair bit of knowledge about closed contours which is why this paper will not look into the proof. Instead of proving the theorem, the paper will show that this equation does work by using it for special cases of  $f(z)$ . The examples shown in 'Few Examples' will be used as  $f$ , and for simplicity, the closed contour or ( $\Gamma$ ) will be the unit circle in both examples.

### Few Examples

**Example 3.** The integral of  $f(z) = z^n$  for  $n > 0$  over the unit circle ( $\gamma$ ) is zero. This problem can be solved by Cauchy's Theorem if it is proven that  $f$  is holomorphic. Since the unit circle is a closed contour. Since it is already proven that  $f$  is holomorphic it implies that the integral is equal to zero. This is why Cauchy's theorem is applicable in many cases, as it just requires knowing whether ( $\gamma$ ) is closed and  $f$  is holomorphic. To check the validity of the method we evaluate the integral of  $f$  on ( $\gamma$ ), where  $f$  is  $z^n$  and ( $\gamma$ ) is the unit circle

$$z(t) = \cos(t) + isin(t) = e^{it}, t \in [0, 2\pi]$$

$$z't = ie^{it}$$

$$f(z(t)) = zn(t) = e^{int}.$$

Now,

$$\begin{aligned} \int_a^b f(z(t))z'(t)dt &= i \int_0^{2\pi} e^{int} e^{it} dt \\ &= i \int_0^{2\pi} e^{i(n+1)t} dt \\ &= \left. \frac{e^{i(n+1)t}}{n+1} \right]_0^{2\pi} \\ &= \left( \frac{e^{i(n+1)2\pi}}{n+1} \right) - \left( \frac{e^{i(n+1)0}}{n+1} \right) \\ &= \left( \frac{1}{n+1} \right) - \left( \frac{1}{n+1} \right) \\ &= 0 \end{aligned}$$

This proves that the integral of  $f$  over  $\gamma$  is equal to zero and that Cauchy's Theorem is true for the special case

**Example 4.** Is the integral of  $f(z) = \underline{z}$  over the unit circle equal to zero? Using Cauchy's theorem, we can say that the integral is not equal to zero because, as proven above,  $f$  is not holomorphic.

$$z(t) = \cos(t) + isin(t) = e^{it}, t \in [0, 2\pi]$$

$$z't = ie^{it} f(z(t)) = e^{\bar{i}t} = e^{-it}.$$

Now,

$$\begin{aligned} \int_a^b f(z(t))z'(t)dt &= i \int_0^{2\pi} e^{\bar{i}t} e^{it} dt \\ &= i \int_0^{2\pi} e^{-it} e^{it} dt \\ &= i \int_0^{2\pi} dt \\ &= 2\pi i \neq 0 \end{aligned}$$

### Example 5 <sup>2</sup>

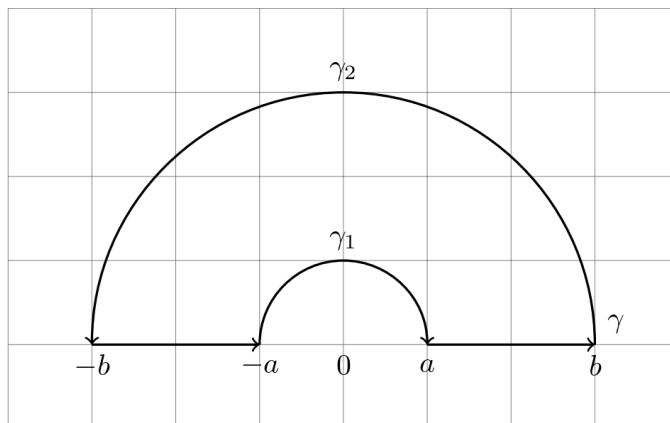
$$\int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

(Hint: The Integral equals  $\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx$  Use indented semi-circle.)

**Proof.** We evaluate this Contour,  $\gamma$  over the function  $f(z) = \frac{e^{iz}}{z}$

We can further split  $\gamma$  into four separate integrals and evaluate them individually. Since  $f$  is holomorphic everywhere except at  $z = 0$ , and  $\gamma$  does not consist  $(0 + 0i)$ , this is an integral over a closed contour over a holomorphic curve, which means that by Cauchy's Theorem

$$\oint_{\gamma} f(z)dz = \int_{-b}^{-a} f(x)dx + \int_a^b f(x)dx + \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = 0$$



We now evaluate the first two integrals

$$\int_{-b}^{-a} \frac{e^{ix}}{x} dx + \int_a^b \frac{e^{ix}}{x} dx$$

Making the substitution  $u = -x$  and  $-du = dx$  in the first integral

$$\begin{aligned} & \int_a^b \frac{e^{ix}}{x} dx + \int_b^a \frac{-e^{-iu}}{-u} du \\ \rightarrow & \int_a^b \frac{e^{ix}}{x} dx - \int_a^b \frac{e^{-iu}}{u} du. \end{aligned}$$

Since  $u$  is a dummy variable, the integral above can be expressed as

$$\begin{aligned} & \int_a^b \frac{e^{ix} - e^{-ix}}{x} dx \\ \Rightarrow & 2i \int_a^b \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{1}{x} dx \\ \Rightarrow & 2i \int_a^b \frac{\sin(x)}{x} dx. \end{aligned}$$

If we take the limit of the integral as  $a \rightarrow 0$  and  $b \rightarrow \infty$  we get the integral in the problem

$$\begin{aligned} \lim_{a \rightarrow 0, b \rightarrow \infty} & 2i \int_a^b \frac{\sin(x)}{x} dx \\ \Rightarrow & 2i \int_0^\infty \frac{\sin(x)}{x} dx. \end{aligned}$$

Now we evaluate the integral over  $\gamma_1$

$$\begin{aligned} z(t) &= ae^{i(\pi-t)}, t \in [0, \pi] \\ z'(t) &= -aie^{i(\pi-t)} dt = -iz(t) dt \end{aligned}$$

Now

$$\int_{\gamma_1} f(z) dz = -i \int_0^\pi \frac{e^{iz(t)}}{z(t)} z'(t) dt$$

$$= -i \int_0^\pi e^{iae^{i(\pi-t)}} dt.$$

As  $a \rightarrow 0$

$$\begin{aligned} &= \lim_{a \rightarrow 0} -i \int_0^\pi e^{iae^{i(\pi-t)}} dt \\ &= \int_0^\pi e^{i0} dt \\ &= -i\pi \end{aligned}$$

Finally, we evaluate the integral over  $\gamma_2$

$$\begin{aligned} z(t) &= be^{it} t \in [0, \pi] \\ z'(t) &= ibe^{it} dt = iz(t) dt \end{aligned}$$

Now:

$$\begin{aligned} \int_{\gamma_2} f(z) dz &= i \int_0^\pi \frac{e^{iz(t)}}{z(t)} z'(t) dt \\ &= i \int_0^\pi e^{ibe^{it}} dt \\ &= i \int_0^\pi e^{ib \cos(t) - b \sin(t)} dx \\ &= i \int_0^\pi e^{ib \cos(t)} e^{-b \sin(t)} dt. \end{aligned}$$

We note that

$$\begin{aligned} |i \int_0^\pi e^{ib \cos(t)} e^{-b \sin(t)} dt| &\leq \int_0^\pi e^{-b \sin(t)} dt \\ &\leq \lim_{b \rightarrow \infty} \int_0^\pi e^{-b \sin(t)} dt \\ &\leq 0 \end{aligned}$$

Since this integral is strictly positive and is greater than or equal to zero, by the squeeze theorem the value of the integral must be zero.

$$\int_{\gamma_2} f(z) dz = i \int_0^\pi e^{ibe^{it}} dt = 0.$$

Now add the three the integrals

$$\begin{aligned} \oint_{\gamma} f(z)dz &= \int_{-b}^{-a} f(x)dx + \int_a^b f(x)dx + \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = 0 \\ &\Rightarrow 2i \int_0^{\infty} \frac{\sin(x)}{x} dx - i\pi + 0 = 0 \\ &\Rightarrow \int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2} \end{aligned}$$

## The Reimann zeta function and the gamma function

The definition of the zeta function, for  $\Re(s) > 1$  is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

The Riemann zeta function can be continued analytically for  $\Re(s) \leq 1$ . This function is the base for important observations such as the prime number theorem. This is a holomorphic function in the half plane and when continued it extends and produces several trivial zeros or roots, which are significant in many other fields – not just mathematics. Finding the non-trivial zeros of the zeta function is an important problem in mathematics and it was proposed by Bernhard Reimann himself.

There are many proofs that the  $\zeta$  function can be analytically continued, or that the function can be extended graphically into the complex plane. See<sup>2</sup>, page 182 for the proof. This section of the paper is about the representation of  $\zeta$  in terms of an integral, which can be used to compute values of  $\zeta(s)$  at different  $s$ .

**Theorem 3.2.** If  $\Re(s) > 1$ , then the zeta function can be expressed as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{u^{s-1}}{e^u - 1} du,$$

where.

$$\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u} du.$$

Proof: This proof modifies  $\Gamma(s)$  to produce  $\zeta(s)$  and the equation is rearranged to make  $\zeta$  the subject. In the definition of the gamma function,

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$$

making, the substitution  $x = nu$ , for  $n \in \mathbb{N}$ , implies that  $dx =$

$ndu$ . Hence, we get:

$$\begin{aligned} \Gamma(s) &= \int_0^{\infty} (nu)^{s-1} e^{-nu} ndu \\ &= \int_0^{\infty} n^{s-1} u^{s-1} (e^{-u})^n n^1 du \\ &= n^s \int_0^{\infty} u^{s-1} (e^{-u})^n du \end{aligned}$$

Now taking the sum from zero to infinity on both sides while dividing both sides by  $n^s$ :

$$\sum_{n=1}^{\infty} \Gamma(s) \cdot \frac{1}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} u^{s-1} (e^{-u})^n du.$$

Since the integral is strictly convergent and positive when applied on the given limits. Thus, the sum and integral can be swapped on the right-hand side. On the left-hand side  $\Gamma(s)$  is independent of  $n$ , which means that  $T(s)$  can be considered as a constant and factored out of the sum. Putting all the observations together yields:

$$\Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} u^{s-1} \sum_{n=1}^{\infty} \left(\frac{1}{e^u}\right)^n du$$

Notice that the left-hand side of the equation is just  $\Gamma(s)\zeta(s)$ , derived from the definition of  $\zeta(s)$ . The infinite series on the right-hand side is a geometric progression, with  $a = r = \frac{1}{e^u}$ . Thus:

$$\begin{aligned} S_{\infty} &= \frac{a}{1-r} \\ &= \frac{\frac{1}{e^u}}{1 - \frac{1}{e^u}} \\ &= \frac{1}{e^u - 1}. \end{aligned}$$

Therefore  $\Gamma(s)\zeta(s)$  can be written as:

$$\begin{aligned} \Gamma(s)\zeta(s) &= \int_0^{\infty} \frac{u^{s-1}}{e^u - 1} du \\ \zeta(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{u^{s-1}}{e^u - 1} du. \end{aligned}$$

## Zeros of the zeta function

This section provides proofs regarding the zeros of the zeta function, where it proves that  $\zeta(s)$  converges and has no zeros for  $\Re(s) > 1$ ,  $\Gamma(s)$  has no zeros for  $\Re(s) > 1$  and that  $\zeta(s)$  only has zeros in  $\{2\alpha \mid \alpha \in \mathbb{Z}^-\}$ . These proofs are important to study the zeta function and its zeros, and to study the critical strip and the critical line. Proposition 4.1. If  $m(s) > 1$ , then the series  $f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$  converges.

Proof:

$$\begin{aligned}
 |K(s)| &= \left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| \\
 &\leq \sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| \\
 &= \sum_{n=1}^{\infty} \left| \frac{1}{n^{\sigma+it}} \right| \\
 &= \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \frac{1}{|n^{it}|} \\
 &= \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}}
 \end{aligned}$$

By Cauchy's Integral theorem, the above sum must converge or diverge if its corresponding integral converges or diverges.

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} &\longleftrightarrow \int_1^{\infty} t^{-\sigma} dt \\
 &= \left[ \frac{t^{1-\sigma}}{1-\sigma} \right]_1^{\infty}
 \end{aligned}$$

For the above expression to converge  $1 - \sigma < 0$ , which implies that  $\sigma > 1$ . Since  $\sigma = \Re(s)$ ,  $\zeta(s)$  must converge when  $\Re(s) > 1$ .

**Theorem 4.2.**  $\Re(z) > 1$  then  $\Gamma(z) = 0$ .

**Proof.** Assume as a contradiction that there exists  $\alpha \in \mathbb{C}$  such that  $\Re(\alpha) > 1$  and  $\Gamma(\alpha) = 0$ . Then for any  $s \geq 0$ :

$$\begin{aligned}
 0 &= \frac{\Gamma(\alpha)}{(s+1)^{\alpha}} \\
 &= \frac{1}{(s+1)^{\alpha}} \int_0^{\infty} t^{\alpha-1} e^{-t} dt.
 \end{aligned}$$

Substituting  $t = (1+s)x$ , it follows that

$$0 = \int_0^{\infty} x^{\alpha-1} e^{-x} e^{-sx} dx$$

Let  $0 < \sigma < \Re(\alpha)$  such that  $\Gamma(\sigma) > 0$ ; such a  $\sigma$  exists by the integral definition of the  $\Gamma$  function. Multiplying above equation by  $\frac{s^{\sigma-1}}{\Gamma(\sigma)}$  and integrating with respect to  $s$  yields

$$\begin{aligned}
 0 &= \int_0^{\infty} \left( \frac{s^{\sigma-1}}{\Gamma(\sigma)} \int_0^{\infty} x^{\alpha-1} e^{-xe^{-n}} dx \right) ds \\
 &= \int_0^{\infty} x^{\alpha-1} e^{-x} \left( \frac{1}{\Gamma(\sigma)} \int_0^{\infty} x^{\sigma-1} e^{-sx} dx \right) dx
 \end{aligned}$$

Substituting  $sx = y$ :

$$\begin{aligned}
 0 &= \int_0^{\infty} x^{\alpha-1} e^{-x} \left( \frac{1}{\Gamma(\sigma)} \int_0^{\infty} \frac{y^{\sigma-1} x^{1-\sigma-\alpha} e^{-y}}{x} dy \right) dx \\
 &= \int_0^{\infty} \frac{x^{\alpha-1}}{x^{\sigma}} e^{-x} \left( \frac{1}{\Gamma(\sigma)} \int_0^{\infty} y^{\sigma-1} e^{-y} dy \right) dx \\
 &= \int_0^{\infty} \frac{x^{\alpha-1}}{x^{\sigma}} e^{-x} \left( \frac{1}{\Gamma(\sigma)} \Gamma(\sigma) \right) dx \\
 &= \int_0^{\infty} x^{(\alpha-\sigma)-1} e^{-x} dx \\
 &\Rightarrow 0 = \Gamma(\alpha - \sigma).
 \end{aligned}$$

Since  $\Gamma$  is a holomorphic function for  $\Re(s) > 1$ ,  $\alpha$  must be an isolated zero<sup>2</sup>. On the other hand, the above argument shows that  $\alpha - \sigma$  is also a zero for every  $0 < \sigma < \Re(\alpha)$ , which implies that  $\alpha$  cannot be an isolated zero. Indeed, by choosing  $\alpha$  to be small enough,  $\alpha - \sigma$  is a zero of  $\Gamma$  which can be made arbitrarily close to  $\alpha$ . This contradicts the fact that  $\alpha$  is an isolated zero.

**Proposition 4.4.** The zeta function  $\zeta(s)$  has no zeros for  $\Re(s) > 1$ .

Proof. If  $\Re(s) > 1$ , then

In the above equation,  $\Lambda(n)$  is the function defined by

$$\Lambda(n) = \begin{cases} \ln(p) & \text{if } p \text{ is a prime} \\ 0 & \text{otherwise} \end{cases}$$

This implies that the summand will be zero if  $n$  is not a power of a prime, which yields,

$$\begin{aligned}
 \zeta(s) &= \exp \left( \sum_p \sum_{m=1}^{\infty} \frac{\ln(n)}{\ln(p^m) \cdot p^{ms}} \right) \\
 &= \exp \left( \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \right).
 \end{aligned}$$

Separating the equation into real and imaginary parts with  $s = \sigma + it$ , i-e,  $p - sm = p - \sigma m$ .  $p - itm$  this yields,

$$\zeta(s) = \exp \left( \sum_p \sum_{m=1}^{\infty} \frac{e^{-itm \ln(p)}}{mp^{\sigma m}} \right).$$

The absolute value of  $\zeta^3$  is given by

$$\begin{aligned} |\zeta(s)| &= \left| \exp \left( \sum_p \sum_{m=1}^{\infty} \frac{e^{-im \ln(p)}}{mp^{\sigma m}} \right) \right| \\ &= \left| \exp \left( \sum_p \sum_{m=1}^{\infty} \frac{\cos(mt \ln(p))}{mp^{\sigma m}} + i \sum_p \sum_{m=1}^{\infty} \frac{\sin(mt \ln(p))}{mp^{\sigma m}} \right) \right| \\ &= \exp \left( \sum_p \sum_{m=1}^{\infty} \frac{\cos(mt \ln(p))}{mp^{\sigma m}} \right). \end{aligned}$$

Using [4.6],

$$\begin{aligned} &|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \\ &= \exp \left( \sum_p \sum_{m=1}^{\infty} \frac{3 + 4 \cos(mt \ln(p)) + \cos(2mt \ln(p))}{mp^{\sigma m}} \right) \end{aligned}$$

Using the identity that  $3 + 4 \cos(\theta) + \cos(2\theta) \geq 0$ , with  $\theta = mt \ln(p)$  implies

$$|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1$$

In particular,  $\zeta(\sigma + it) = 0$  if  $\sigma > 0$ .

**Proposition 4.10.** The only zeros of the zeta function  $\zeta(s)$  with  $\Re(s) < 0$  are the negative even integers  $s = -2, -4, -6, \dots$

Proof. The proof is based upon expressing  $\zeta$  as a functional equation, which is

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s)$$

Above Equation is derived in<sup>2</sup>. For  $\zeta$  to be equal to zero, one of the three parts of the equations must equal zero. Note that  $\pi^{s-\frac{1}{2}}$  cannot equal zero. Also,  $\zeta(1-s)$  is not zero because, as proved above,  $\zeta$  has no zeros for  $\Re(s) > 1$ . Further,  $\Gamma(\frac{1-s}{2})$  is also never equal to zero. Hence, the zeros of  $\zeta$  with  $\Re(s) < 0$  coincide with the poles of  $\Gamma(\frac{s}{2})$ .  $\Gamma(s)$  has simple poles at the negative integers, including zero; see Section 6.1.2 of<sup>2</sup>.

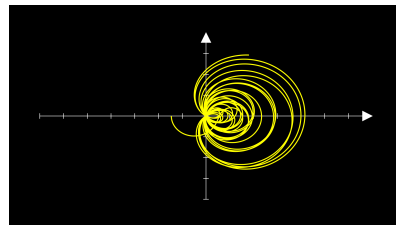
## Numerical Analysis of the zeros

This section of the paper presents a program the author wrote to compute the values of the zeta function on the critical line. The program provides a list of the non-trivial zeros of the zeta function by using the identities and theorems derived above.

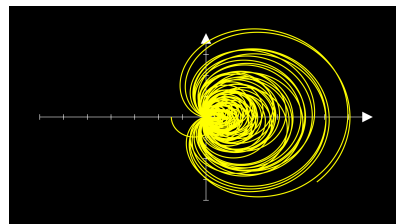
## Computing the zeros of $\zeta$

It is hypothesized that all the zeros of the zeta function lie on the Critical Line. In other words, the zeta function has zeros where the Real part of the input is  $\frac{1}{2}$ . The program in Appendix A computes zeta on the critical strip. All it essentially does is that it computes  $\frac{1}{2} + it$  for  $t \in [0, 1000]$ . Then the Real and Imaginary parts of the output are stored in different lists, namely,  $x\_vals$  and  $y\_vals$ . Then the program checks if the distance of the output from the origin is less than 0.01. If this condition is met then that specific output is added to another list, which is called zeros. The elements in zeros are the zeros of the zeta function. After which the program uses Manim<sup>4</sup> to make a plot of the values obtained. A line graph is plotted, with the x values being the real part of the out and the y values being the imaginary part of the output. This Plot shows how the distribution of the zeros is along the critical line, and the zeta zeros are where the line intercepts the origin.

To calculate the values of the zeta function, the program uses a library called MpMath<sup>3</sup>. The Library has inbuilt functions to calculate the zeta function and other functions related to it, such as the gamma function. The documentation provides detailed information about how it calculates these functions. The definitions used are in line with that shown in the paper, and thus this library was used. The program also uses NumPy<sup>5</sup>, which is a library that provides various functions including the ones used in complex Analysis. This is essentially



the plot obtained after computing values for the zeta function on the critical line. The horizontal axis is the  $\Re$  axis and the vertical axis is the  $\Im$  axis. In the simulation above, a zeta-zero is essentially when the yellow line intercepts the origin. These values are noted down by the program and stored, the obtained values are listed in Appendix A. The images shown below are the same simulation, but the domain of the  $\Im$  part is larger.



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## Uncertainties in the Program

The accuracy of the program is until 2 decimal places for the output of each input. Which is why some roots may be incorrect or repeated. A repeated root is two numbers that are in close vicinity to each other, which indicate the same root but due to the accuracy set in the program, the zero is listed as two different numbers. An example of this is 14.13 and 14.14, both these numbers are not roots of the zeta function, but are close approximations of the actual root, that lies somewhere between these two numbers.

The uncertainty of the program can be fixed by taking into consideration more decimal places when computing the values. The accuracy of the results will improve when the order of magnitude of the output becomes more accurate.

## Appendix A. Source code

zeros = [14.13, 14.14, 21.02, 21.03, 25.01, 30.42, 30.43, 32.93, 32.94, 37.59, 40.92, 43.33, 48.00, 48.01, 49.77, 49.78, 52.97, 56.45, 59.34, 59.35, 60.83, 65.11, 67.08, 69.55, 72.07, 75.70, 75.71, 77.14, 77.15, 79.34, 82.91, 84.74, 87.42, 87.43, 88.81, 92.49, 94.65, 95.87, 98.83, 101.32, 103.73, 105.45, 107.17, 111.03, 111.87, 111.88, 114.32, 118.79, 121.37, 122.95, 124.26, 129.58, 131.09, 133.50, 134.76, 139.74, 141.12, 143.11, 146.00, 147.42, 150.05, 150.92, 150.93, 156.11, 157.60, 158.85, 161.19, 163.03, 165.54, 169.09, 169.10, 169.91, 173.41, 174.75, 176.44, 178.38, 184.87, 184.88, 185.60, 187.23, 192.03, 193.08, 196.88, 202.49, 204.19, 211.69, 213.35, 214.55, 216.17, 219.07, 220.71, 220.72, 221.43, 224.01, 224.98, 227.42, 229.34, 231.25, 231.99, 237.77, 241.05, 242.82, 244.07, 247.14, 248.10, 249.57, 253.07, 256.38, 258.61, 259.87, 259.88, 265.56, 266.61, 266.62, 267.92, 269.97, 273.46, 275.59, 276.45, 278.25, 279.23, 283.21, 286.67, 287.91, 289.58, 293.56, 294.96, 294.97, 295.57, 297.98, 299.84, 301.65, 302.70, 304.86, 305.73, 307.22, 310.11, 312.43, 321.16, 322.14, 323.47, 329.03, 329.95, 333.65, 334.21, 338.34, 339.86, 341.04, 346.35, 347.27, 350.41, 351.88, 353.49, 356.02, 357.15, 357.95, 361.29, 363.33, 367.99, 368.97, 370.05, 373.06, 375.82, 375.83, 376.32, 376.33, 384.96, 385.86, 392.25, 393.43, 395.58, 396.38, 397.92, 401.84, 402.86, 404.24, 407.58, 411.97, 413.26, 415.02, 415.45, 415.46, 419.86, 420.64, 425.07, 427.21, 428.13, 430.33, 431.30, 432.14, 433.89, 436.16, 437.58, 438.62, 439.92, 444.32, 446.86, 447.44, 449.15, 450.13, 456.33, 459.51, 460.09, 465.67, 466.57, 467.44, 472.80, 475.60, 476.77, 478.94, 481.83, 483.85, 485.54, 486.53, 488.38, 489.66, 491.40, 493.31, 493.96, 495.36, 496.43, 498.58, 500.31, 501.60, 502.28, 504.50, 508.80, 511.56, 512.62, 513.67, 517.59, 522.46, 523.96, 527.90, 528.41, 529.81, 532.69, 533.78, 537.07, 538.43, 540.21, 540.22, 540.63, 540.64, 547.01,

547.93, 550.97, 552.05, 555.79, 556.90, 557.56, 557.57, 560.24, 562.56, 564.16, 564.50, 564.51, 566.70, 567.73, 570.05, 572.42, 575.09, 575.81, 577.04, 579.10, 584.56, 586.74, 588.14, 590.66, 591.73, 592.57, 595.73, 596.36, 601.60, 602.58, 603.63, 609.39, 610.84, 613.60, 615.54, 620.27, 621.71, 624.27, 626.02, 627.27, 630.47, 630.48, 630.80, 630.81, 637.40, 637.92, 637.93, 638.93, 643.28, 644.99, 646.35, 647.76, 648.79, 650.20, 650.67, 653.65, 654.30, 655.71, 665.34, 666.51, 666.52, 667.15, 672.46, 673.04, 676.14, 677.23, 677.80, 682.60, 684.01, 684.97, 689.37, 692.45, 693.18, 700.30, 701.30, 702.23, 708.27, 709.23, 711.13, 711.90, 712.75, 718.74, 721.35, 722.28, 728.40, 728.41, 728.76, 734.79, 738.58, 739.91, 740.57, 746.50, 747.67, 747.68, 748.24, 750.66, 750.97, 755.84, 756.77, 758.10, 758.90, 762.70, 763.59, 764.31, 767.22, 768.28, 771.07, 772.96, 774.12, 775.05, 776.00, 777.30, 780.35, 782.14, 782.60, 784.29, 785.74, 786.46, 787.47, 790.06, 790.83, 792.43, 792.89, 802.54, 803.24, 809.20, 810.08, 812.77, 814.87, 818.38, 820.72, 821.71, 821.72, 822.20, 826.04, 828.34, 831.80, 836.69, 837.35, 838.25, 842.04, 847.97, 848.49, 849.86, 857.31, 860.41, 861.17, 863.19, 864.34, 866.42, 868.67, 872.19, 873.10, 873.91, 876.60, 879.38, 883.43, 884.20, 885.27, 893.12, 894.89, 895.40, 899.22, 899.86, 900.85, 902.24, 903.10, 905.83, 912.33, 912.82, 914.73, 918.84, 919.45, 922.50, 927.85, 928.66, 931.01, 936.23, 939.66, 946.76, 946.77, 947.07, 947.08, 950.15, 954.13, 954.83, 957.51, 959.46, 961.67, 963.18, 963.57, 966.11, 967.37, 971.07, 973.87, 976.18, 980.58, 981.29, 986.13, 992.73, 993.21, 993.22, 999.79]

```
from manim import *
import mpmath as mp
import numpy as np

strip_vals=[]
x_vals=[]
y_vals=[]
zeros=[]
for i in range(0,1000000):
    k=i/1000
    strip_vals.append(k)
for i in range(len(strip_vals)):
    l=mp.zeta(complex(0.5,strip_vals[i]))
    y_vals.append(np.imag(l))

for i in range(len(strip_vals)):
    l=mp.zeta(complex(0.5,strip_vals[i]))
    x_vals.append(np.real(l))

print(strip_vals)
print(x_vals)
print(y_vals)
print(len(zeros))
for i in range(len(x_vals)):
    if np.abs(complex(x_vals[i],y_vals[i]))<0.01:
        zeros.append(strip_vals[i])

with open(r'D:\Arkit\Portfolio
2020\Math\Lumiere\Fellowship\Codes\strip_vals.txt', 'w') as f:
    print(zeros,file=f)

class Zeta(Scene):
    def construct(self):
        n=Axes()
        p=n.plot_line_graph(x_vals,y_vals,add_vertex_dots=False)
        self.add(n)
        self.play(Create(p),run_time=100,rate_function=linear)
```

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